Math 213 - Linear Approximation and Error

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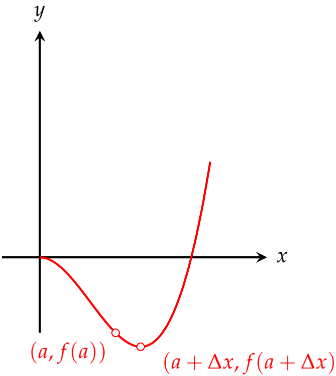
October 2, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

Approximation in Calculus I-II



Suppose

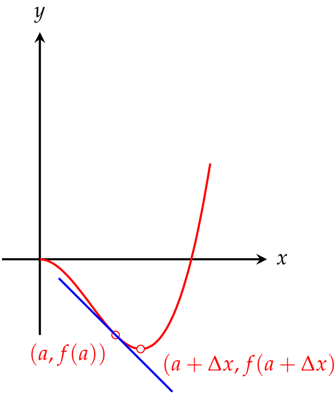
$$f(x) = x^3 - 2x^2,$$
$$f'(x) = 3x^2 - 4x,$$
$$f''(x) = 6x - 4$$

and

$$a = 1, \quad \Delta x = 1/3$$

How can we approximate $f(a + \Delta x)$?

Approximation in Calculus I-II



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How can we approximate $f(a + \Delta x)$?

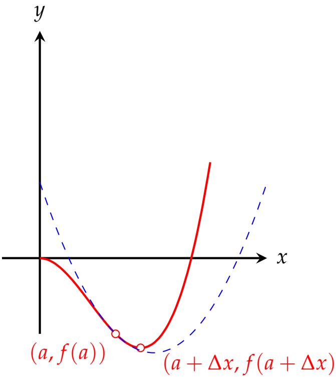
First try: Linear Approximation

$$f(a + \Delta x) \sim f(a) + f'(a) \cdot \Delta x$$

$$f(1 + \Delta x) \sim \underbrace{-1}_{f(1)} + \underbrace{(-1)}_{f'(1)} \cdot (\Delta x)$$

What's wrong?

Approximation in Calculus I-II



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How can we approximate $f(a + \Delta x)$?

Second try: Quadratic Approximation

$$f(a + \Delta x) = f(a) + f'(a)\Delta x + \frac{f''(a)}{2}(\Delta x)^2$$

$$f(1 + \Delta x) = \underbrace{-1}_{f(1)} + \underbrace{-1}_{f'(1)} \cdot \Delta x + \frac{1}{2} \underbrace{2}_{f''(1)} \cdot (\Delta x)^2$$

Big Ideas of the Day - Formulas

For a function $f(x)$ of one variable, the quadratic approximation to $f(x)$ near $x = a$ is:

$$f(a + \Delta x) \sim f(a) + f'(a)\Delta x + \frac{f''(a)}{2}(\Delta x)^2$$

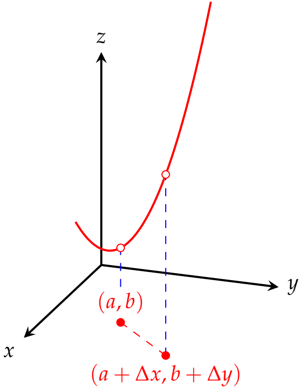
For a function of $f(x, y)$ of two variables, we can approximate $f(x, y)$ near $(x, y) = (a, b)$ by quadratic approximation to the function

$$F(t) = f(a + t\Delta x, b + t\Delta y)$$

because

$$F(0) = f(a, b)$$
$$F(1) = f(a + \Delta x, b + \Delta y)$$

Big Ideas of the Day - Pictures



Let

$$F(t) = f(x(t), y(t))$$

$$x(t) = a + t\Delta x$$
$$y(t) = b + t\Delta y$$

Since

$$F(0) = f(a, b), \quad F(1) = f(a + \Delta x, b + \Delta y),$$

we can approximate $f(a + \Delta x, b + \Delta y)$ by applying quadratic approximation to $F(t)$.

Approximation for Functions of Several Variables

Given a function $f(x,y)$ and a point (a,b) where we know $f(a,b)$ and its partial derivatives, how can we find $f(a+\Delta x, b+\Delta y)$?

Idea of the Day: Let

$$F(t) = f(a + t\Delta x, b + t\Delta y)$$

Notice that:

- $F(0) = f(a,b)$ and $g(1) = f(a+\Delta x, b+\Delta y)$
- $F'(0) = \frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y$
- $F''(0) = \frac{\partial^2 f}{\partial x^2}(a,b)(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(\Delta x \Delta y) + \frac{\partial f^2}{\partial y^2}(a,b)(\Delta y)^2$

We will check these formulas in class!

From

$$F(t) \sim F(0) + F'(0)\cdot t + \frac{1}{2}F''(0)t^2$$

with $t = 1$, we can get an approximation of $f(a+\Delta x, b+\Delta x)$

Big Result of the Day

$$f(a+\Delta x, y+\Delta y) \sim f(a,b)$$

$$+ \underbrace{\frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y}_{\text{linear approximation}}$$

$$+ \underbrace{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(\Delta x \Delta y) + \frac{\partial f^2}{\partial y^2}(a,b)(\Delta y)^2\right)}_{\text{quadratic correction}}$$

Puzzler # 1

$$f(a + \Delta x, y + \Delta y) \sim f(a, b) + \frac{\partial f}{\partial x}(a, b)\Delta x + \frac{\partial f}{\partial y}(a, b)\Delta y$$

Find the linear approximation to $f(x, y) = \sqrt{x^2 + y^2}$ at $(3, 4)$ and use it to approximate $\sqrt{(3.01)^2 + (3.98)^2}$

Error Analysis

Suppose we are approximating a quantity Q and make an error ΔQ.

"Error" comes in (at least) three flavors.

- *Absolute error*: ΔQ
- *Relative error*: $\dfrac{\Delta Q}{Q}$
- *Percentage error*: $100\dfrac{\Delta Q}{Q}$

We'll show how to use linear approximation to estimate the error made in computing a quantity Q from input measurements with error.

Puzzler #2

We measure the sides of a box as x, y, z with some percentage error. Secretly, the box's sides are $x = 10$, $y = 20$, $z = 20$ in appropriate units.

(1) If we make a 2% error in each measurement, what is the maximum percentage error we make in computing the volume from our measurements?

(2) How accurate do our measurements have to be to compute $V(x,y,z) = xyz$ with no more than 1% error?

$$V(x+\Delta x, y+\Delta y, z+\Delta z) \sim V(x,y,z) + \frac{\partial V}{\partial x}(x,y,z)\Delta x + \frac{\partial V}{\partial y}(x,y,z)\Delta y + \frac{\partial V}{\partial z}(x,y,z)\Delta z$$

Puzzler #3

The volume of a cylinder having radius r and height h is $V = \pi r^2 h$. What is the percentage increase in V is r is increased by 2% and h is increased by 2%?

Quadratic Approximation

Find the quadratic approximation to $f(x,y) = e^{2x}\sin(3y)$ at $(x,y) = (0,0)$ by cheating.[1]

[1] To be clear, what I mean here is finding an easy shortcut to the answer. Actual cheating is, of course, forbidden by University regulations.

Reminders for the Week of October 2-6

- Homework B4 due on Wednesday 10/4 at 11:59 PM
- Quiz #5 on higher-order derivatives, chain rule, and tangent planes due at 11:59 PM