Math 213 - Directional Derivatives and the Gradient

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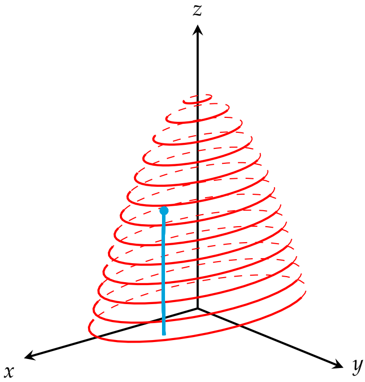
October 4, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

Problem of the Day



Water always flows downhill in the steepest direction. Suppose that the height in feet near a certain mountaintop is given by

$$f(x,y) = 2000 - x^2 - 2y^2$$

A stream of water starts at $(x,y) = (1, \frac{1}{2})$. What path does it take downhill?

Problem: How do we find which direction is the steepest one at each point?

Adapted from Libre Texts

Directional Derivatives

Let's find out:

- How to find the rate of change of f at (a, b) in any direction $\mathbf{u}$, where $\mathbf{u}$ is a unit vector
- How to find the direction of maximum change at (a, b)

if $\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j}$, the rate of change we want is $h'(0)$ where

$$h(t) = f(a + tu_x, b + tu_y)$$

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We get

$$h'(0) = \frac{\partial f}{\partial x}(a, b)u_x + \frac{\partial f}{\partial y}(a, b)u_y$$

or

$$h'(0) = \left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle \cdot \mathbf{u}$$

Directional Derivatives

If (a, b) is a point and $\mathbf{u}$ is a unit vector, the directional derivative of f at (a, b) in the direction $\mathbf{u}$ is given by

$$D_{\mathbf{u}} f(a, b) = (\nabla f)(a, b) \cdot \mathbf{u}$$

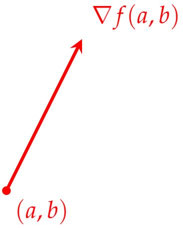
where

$$\nabla f(a, b) = \left\langle \frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b) \right\rangle$$

is the *gradient* of f at (a, b).

Find the gradient of $f(x, y) = 2000 - x^2 - 2y^2$ and use it to find the directional derivative of $f(x, y)$ at $(1, \frac{1}{2})$ in the direction of the vector $\mathbf{i} + \mathbf{j}$.

The Gradient - Big Idea of the Day



The directional derivative of f at (a,b) in the direction $\mathbf{u}$ is

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The directional derivative is:

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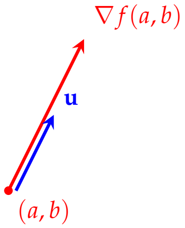

The directional derivative of f at (a, b) in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}} f(a,b) = (\nabla f)(a,b) \cdot \mathbf{u}$$

The directional derivative is:

- *most positive* when $\mathbf{u}$ is in the direction of $(\nabla f)(a, b)$,

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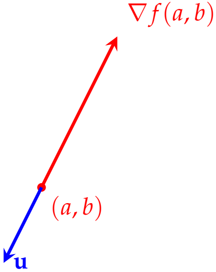

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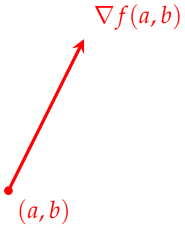
The directional derivative is:

- *most positive* when $\mathbf{u}$ is in the direction of $(\nabla f)(a,b)$,
- *most negative* when $\mathbf{u}$ points in the opposite direction from $(\nabla f)(a,b)$

The Gradient - Big Idea of the Day



The directional derivative of f at (a,b) in the direction $\mathbf{u}$ is

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The Gradient - Big Idea of the Day

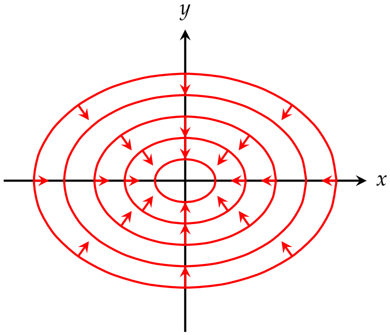
The gradient

$$(\nabla f)(a,b) = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle$$

is the true "first derivative" of a function of two variables.

- $(\nabla f)(a,b)$ points in the direction of greatest increase of f at (a,b)
- $-(\nabla f)(a,b)$ points in the direction of greatest decrease.
- $|(\nabla f)(a,b)|$ is the maximum rate of change of f at (a,b).

Look - Here's Proof!



At left is a contour map for the function

$$f(x) = 2000 - (x^2 + 2y^2)$$

showing the gradient vector at various points

For this function,

$$(\nabla f)(x, y) = \langle -2x, -4y \rangle$$

How is the direction of the gradient related to the level curves of f?

The Gradient Also Works for Functions of Three Variables

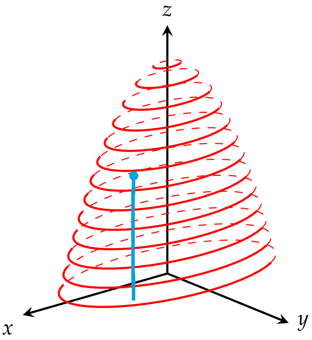
If $g(x, y, z)$ is a function of three variables, we define

$$(\nabla g)(a, b, c) = \left\langle \frac{\partial g}{\partial x}(a, b, c), \frac{\partial g}{\partial y}(a, b, c), \frac{\partial g}{\partial z}(a, b, c) \right\rangle$$

The gradient points in the direction of maximum increase of g, and that rate of increase is $|(\nabla g)(a, b, c)|$

Find the direction of greatest change of the function $g(x, y, z) = x^2 + 2y^2 + 4z^2$ at $(x, y, z) = (2, 1, 2)$.

Falling Water Revisited



Recall the height function for the mountain

$$f(x,y) = 2000 - x^2 - 2y^2$$

Its gradient is

$$(\nabla f)(x,y) = \langle -2x, -4y \rangle$$

so the water follows

$$-(\nabla f)(x,y) = \langle 2x, 4y \rangle$$

starting at $(x,y) = (1, \frac{1}{2})$

If the path of the water is given by $(x(t), y(t), z(t))$ then

$$\begin{aligned} \dot{x}(t) &= 2x(t) \\ \dot{y}(t) &= 4y(t) \\ z(t) &= f(x(t), y(t)) \end{aligned}$$

Adapted from Libre Texts

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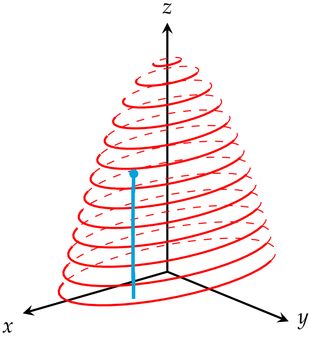


$$\dot{x}(t) = 2x(t)$$
$$\dot{y}(t) = 4y(t)$$
$$z(t) = f(x(t), y(t))$$

where $(x(0), y(0)) = (1, \frac{1}{2})$

You can solve these differential equations and find that

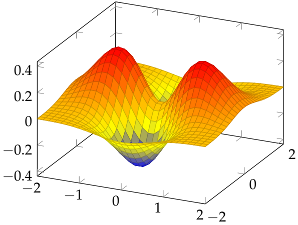
$$x(t) = e^{2t}, \quad y(t) = \frac{1}{2}e^{4t}$$

so the curve it travels is

$$y = \frac{1}{2}x^2, \quad z = 2000 - x^2 - 2y^2$$

Adapted from Libre Texts

Gradient Descent



Find a local minimum of the function

$$f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)}$$

We'll start with a guess and use

$$(\nabla f)(x,y) = \Big\langle -2x(x^2 - y^2 - 1)e^{-(x^2+y^2)}, \; 2y(-x^2 + y^2 - 1)e^{-(x^2+y^2)} \Big\rangle$$

to move in the direction of greatest decrease

Homework: Google "Gradient Descent" and see what links come up!

Reminders for the Week of October 2-6

- Homework B4 due on Wednesday 10/4 at 11:59 PM
- Quiz #5 on higher-order derivatives, chain rule, and tangent planes due at 11:59 PM