Math 213 - Vectors

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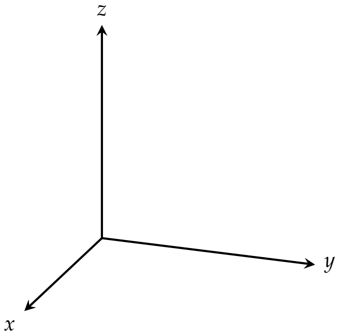
August 23, 2023

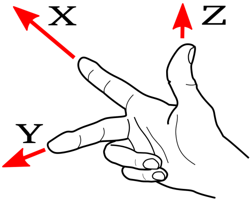
Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

Right-Handed Coordinate Systems

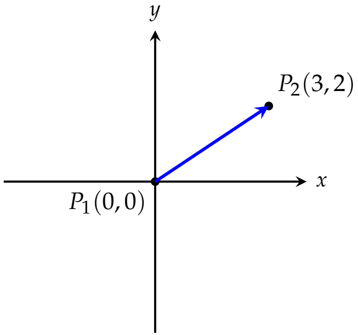
The *xyz* coordinate system we are using is a *right-handed* coordinate system as the pictures below show:

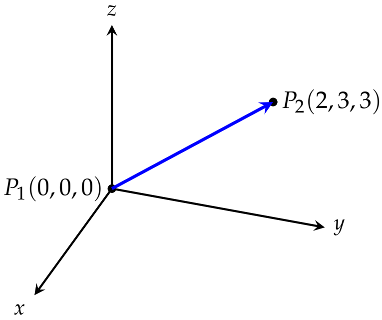




Vectors

A *vector* is a quantity with magnitude and direction. It specifies a displacement from one point to another.





Vectors

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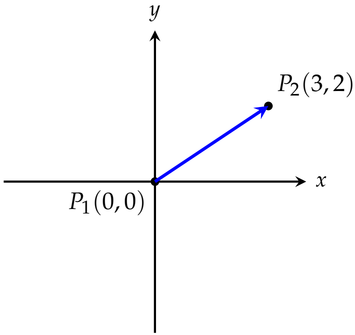
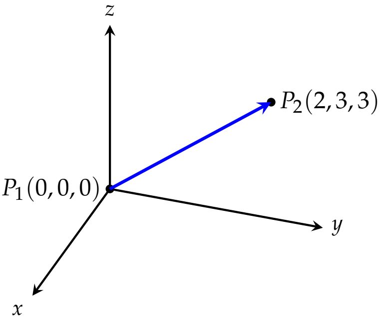


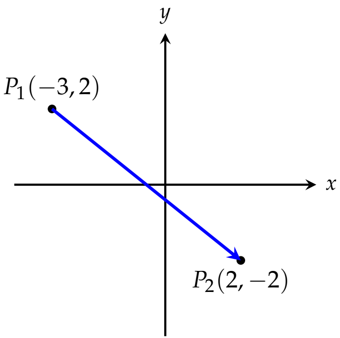
$$\overrightarrow{P_1P_2} = \langle 3, 2 \rangle$$

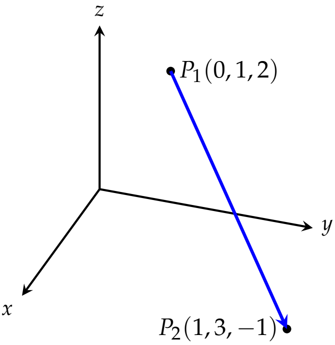


$$\overrightarrow{P_1P_2} = \langle 2, 3, 3 \rangle$$

Vectors

A vector can also specify a displacement from one *nonzero* point to another





$\overrightarrow{P_1P_2} = \langle \quad , \quad \rangle$

$\overrightarrow{P_1P_2} = \langle \quad , \quad , \quad \rangle$

$\overrightarrow{P_1P_2} = \langle 5, -4 \rangle$

$\overrightarrow{P_1P_2} = \langle 1, 2, -3 \rangle$

Vector Notation

Vectors in two dimensions are denoted $\mathbf{v} = \langle a, b \rangle$ where:

a is the displacement in the x direction
b is the displacement in the y direction

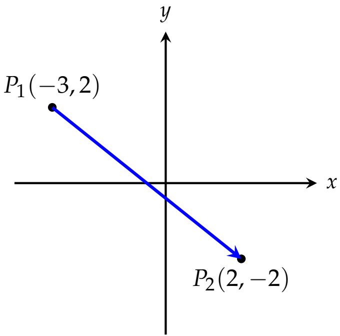
Vectors in three dimensions are denoted $\mathbf{w} = \langle a, b, c \rangle$ where:

a is the displacement in the x direction
b is the displacement in the y direction
c is the displacement in the z direction

Pro Tip: A point in the xy plane is denoted (a, b). A vector in the xy plane is denoted $\langle a, b \rangle$.

Similarly, a point in the xyz plane is denoted (a, b, c). A vector in the xyz plane is denoted by $\langle a, b, c \rangle$.

Vector Notation



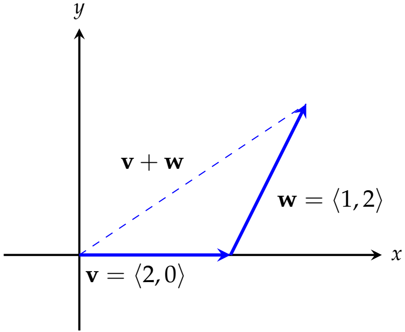
In this picture, the vector

$$\mathbf{v} = \langle 5, -4 \rangle$$

goes from the point $P_1(-3,2)$ to the point $P_2(2,-2)$

Vector Addition

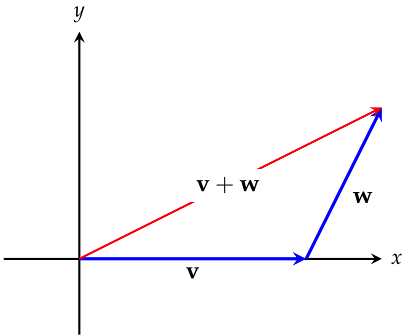
If $\mathbf{v}$ and $\mathbf{w}$ are vectors, representing displacements, the net result of those two displacements is the *vector sum* $\mathbf{v} + \mathbf{w}$.



What is $\mathbf{v} + \mathbf{w}$?
$\langle 3, 2 \rangle$

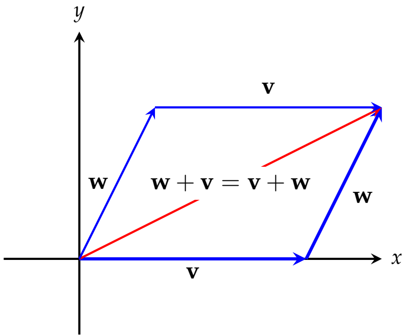
What is $\mathbf{w} + \mathbf{v}$?
$\langle 3, 2 \rangle$

The Parallelogram Law of Vector Addition

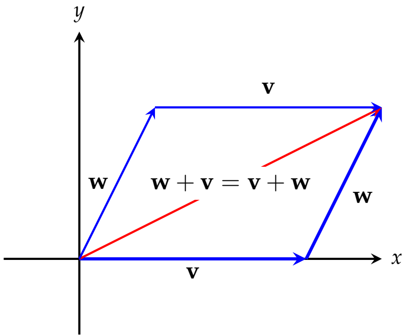


This diagram shows the vector sum $\mathbf{v} + \mathbf{w}$. What happens if we reverse the order?

The Parallelogram Law of Vector Addition



The Parallelogram Law of Vector Addition



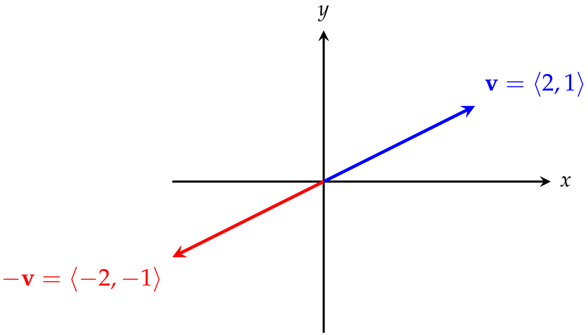
We see that *vector addition commutes*, that is

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

for any vectors $\mathbf{v}$ and $\mathbf{w}$

Vector Subtraction

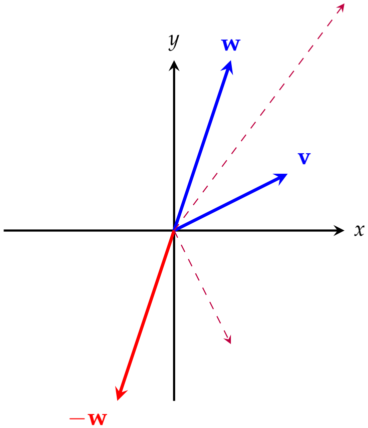
If $\mathbf{v}$ is a vector, then $-\mathbf{v}$ is a vector with the same magnitude but in the opposite direction



The difference of two vectors is

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$

Vector Subtraction



Which of the two dashed vectors is $\mathbf{v}+\mathbf{w}$?

Which is $\mathbf{v}-\mathbf{w}$?

Find $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ if

$$\mathbf{v} = \langle 2, 1 \rangle$$

and

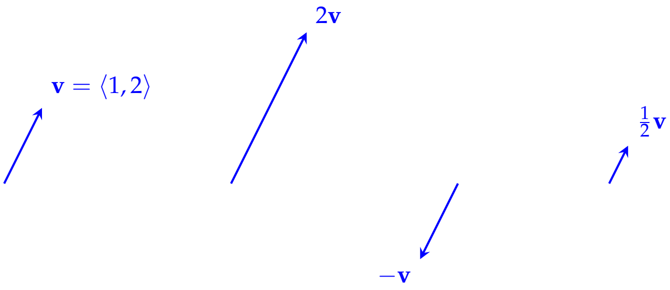
$$\mathbf{w} = \langle 1, 3 \rangle$$

$$\mathbf{v}+\mathbf{w} = \langle 2+1, 1+3 \rangle = \langle 3, 4 \rangle$$
$$\mathbf{v}-\mathbf{w} = \langle 2-1, 1-3 \rangle = \langle 1, -2 \rangle$$

Scalar Multiplication

We can multiply a vector **v** by a scalar (that is, a real number) s and get a new vector cv.

$$s\langle a, b\rangle = \langle sa, sb\rangle$$
$$s\langle a, b, c\rangle = \langle sa, sb, sc\rangle$$



$2\mathbf{v} = \langle 2, 4\rangle \qquad -\mathbf{v} = \langle -1, -2\rangle \qquad \frac{1}{2}\mathbf{v} = \langle \frac{1}{2}, 1\rangle$

Addition and Scalar Mutiplication

If $\mathbf{v} = \langle 2, -1, 3 \rangle$ and $\mathbf{w} = \langle -1, 0, 1 \rangle$, find:

1. $2\mathbf{v} + 3\mathbf{w}$
2. $\mathbf{v} - 3\mathbf{w}$

$$\begin{aligned} 2\mathbf{v} + 3\mathbf{w} &= \langle 4, -2, 6 \rangle + \langle -3, 0, 3 \rangle && \text{scalar multiply first!} \\ &= \langle 1, -2, 9 \rangle && \text{add components} \end{aligned}$$

$$\begin{aligned} \mathbf{v} - 3\mathbf{w} &= \langle 2, -1, 3 \rangle - \langle -3, 0, 3 \rangle && \text{scalar multiply first} \\ &= \langle 2, -1, 3 \rangle + \langle 3, 0, -3 \rangle && \text{multiply by } -1 \\ &= \langle 5, -1, 0 \rangle && \text{add components} \end{aligned}$$

Properties of Addition and Scalar Multiplication

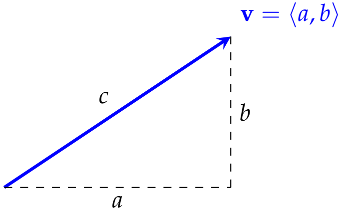
$\mathbf{0}$ is the *zero vector* ($< 0, 0 >$ or $< 0, 0, 0 >$)

(1)	$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	(2)	$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
(3)	$\mathbf{a} + \mathbf{0} = \mathbf{a}$	(4)	$\mathbf{a} + (-\mathbf{a}) = 0$
(5)	$s(\mathbf{a} + \mathbf{b}) = s\mathbf{a} + s\mathbf{b}$	(6)	$(s + t)\mathbf{a} = s\mathbf{a} + t\mathbf{a}$
(7)	$(st)\mathbf{a} = s(t\mathbf{a})$	(8)	$1\mathbf{a} = \mathbf{a}$

These properties are about the...

(1)	commutative law	(2)	associative law (+)
(3)	additive identity	(4)	additive inverse
(5)	distributive law	(6)	distributive law
(7)	associative law ($\times$)	(8)	scalar identity

Length of a Vector



The length of $\mathbf{v}$, denoted $|\mathbf{v}|$, is the length of the hypoteneuse c so

$$|\mathbf{v}| = \sqrt{a^2 + b^2}$$

If

$$\mathbf{v} = \langle 3, 1 \rangle$$

what are the lengths of $\mathbf{v}$, $2\mathbf{v}$, $-2\mathbf{v}$, and $\frac{1}{2}\mathbf{v}$?

What is the length of a vector $\mathbf{v} = \langle a, b, c \rangle$?

$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$

Puzzler #1

Suppose $\mathbf{a} = \langle -1, 0, 1 \rangle$. What is $|\mathbf{a}|$?

What is $\mathbf{b} = \frac{1}{|\mathbf{a}|}\mathbf{a}$?

What is the length of $\mathbf{b}$?

$$|\mathbf{a}| = \sqrt{1+0+1} = \sqrt{2}$$

$$\mathbf{b} = \frac{1}{\sqrt{2}}\langle -1, 0, 1 \rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

Puzzler #1

Suppose $\mathbf{a} = \langle -1, 0, 1 \rangle$. What is $|\mathbf{a}|$?

What is $\mathbf{b} = \dfrac{1}{|\mathbf{a}|}\mathbf{a}$?

What is the length of $\mathbf{b}$?

$$|\mathbf{a}| = \sqrt{1+0+1} = \sqrt{2}$$

$$\mathbf{b} = \frac{1}{\sqrt{2}}\langle -1, 0, 1 \rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

Moral: If $\mathbf{a}$ is a nonzero vector, the vector

$$\mathbf{b} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

is a vector of length 1 in the same direction as $\mathbf{a}$

Unit Vectors

A *unit vector* is a vector with length 1.

Suppose that $\mathbf{v} = \langle 2, -1, 2 \rangle$. Find a unit vector in the direction of $\mathbf{v}$.

First,

$$|\mathbf{v}| = \sqrt{2^2 + 1^2 + 2^2} = 3$$

Next, a unit vector in the direction of $\mathbf{v}$ is

$$\frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{3}\langle 2, -1, 2 \rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

Unit Vectors

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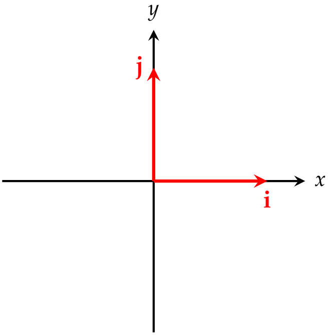

The vectors

$$\mathbf{i} = \langle 1, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1 \rangle$$

are standard basis vectors in two dimensions

Unit Vectors

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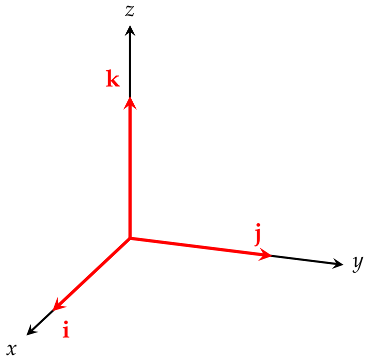
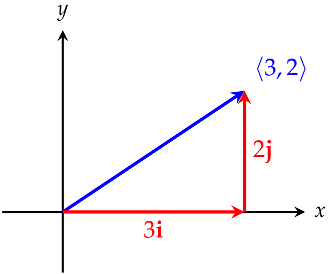


The vectors

$$\begin{aligned} \mathbf{i} &= \langle 1, 0, 0 \rangle \\ \mathbf{j} &= \langle 0, 1, 0 \rangle \\ \mathbf{k} &= \langle 0, 0, 1 \rangle \end{aligned}$$

are standard basis vectors in three dimensions

Representing Vectors



Using the unit vectors

$$\mathbf{i} = \langle 1, 0 \rangle, \quad \mathbf{j} = \langle 0, 1 \rangle$$

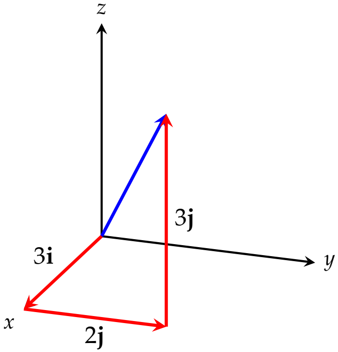
we can represent any vector in the plane as a sum:

$$< 3, 2 >= 3\mathbf{i} + 2\mathbf{j}$$

or more generally

$$< a, b >= a\mathbf{i} + b\mathbf{j}$$

Representing Vectors



Using the unit vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\mathbf{k} = \langle 0, 0, 1 \rangle$$

we can represent any vector in space as a sum:

$$\langle 3, 2, 3 \rangle = 3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

or more generally

$$\langle a, b, c \rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Parallel Vectors

Two vectors $\mathbf{a}$ and $\mathbf{b}$

- are *parallel* if $\mathbf{a} = s\mathbf{b}$ for some nonzero real number s
- are *in the same direction* if $\mathbf{a} = s\mathbf{b}$ for some number $s > 0$

Suppose that

$$\mathbf{v} = \langle 2, -1, 3 \rangle.$$

Which of the following vectors are parallel to $\mathbf{v}$? Which are in the same direction?

$\mathbf{a} = \langle -2, 1, -3 \rangle$	parallel
$\mathbf{b} = \langle 1, 0, -2 \rangle$	not parallel
$\mathbf{c} = \langle 6, -3, 9 \rangle$	parallel, same direction

Parallel Vectors

Suppose that $\mathbf{v} = \langle 2, -2, 2 \rangle$.

What are all the unit vectors parallel to $\mathbf{v}$?

What are all the unit vectors in the same direction as $\mathbf{v}$?

Since $|\mathbf{v}| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$, a unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{1}{2\sqrt{3}}\langle 2, -2, 2 \rangle = \left\langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

This vector points in the same direction as $\mathbf{v}$.
Another vector parallel to $\mathbf{v}$ but in the opposite direction is

$$-\mathbf{u} = \left\langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

These two vectors are the only unit vectors parallel to $\mathbf{v}$.

Reminders for the Week of August 21-25

- Thursday 8/24 - Recitation on CLP3 1.2-Vectors
- Friday 8/25 - Read CLP3 1.2 on Dot Products before class
- Friday 8/25 - Webwork A1 due at 11:59 PM