Math 213 - Local Maxima and Minima

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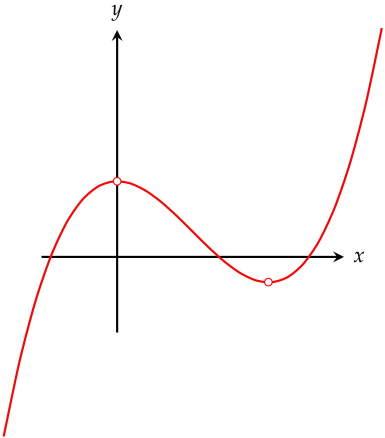
October 6, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

Review - Local Maxima and Minima



A function $y = f(x)$ has a *local minimum* at $x = a$ if $f(a) \leq f(x)$ for all x sufficiently close to a

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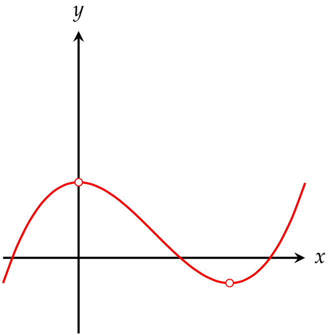
Review - Local Maxima and Minima



At left is the graph of

$$f(x) = \frac{1}{3}x^3 - x^2$$

Note that

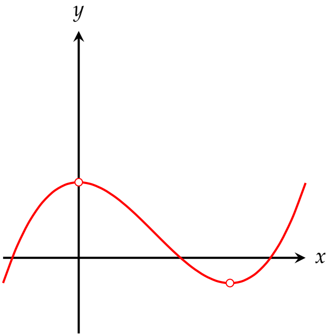
$$f'(x) = x^2 - 2x$$
$$f''(x) = 2x - 2$$

Where are the critical points of f?

Where does f have a local maximum?

Where does f have a local minimum?

Review - Local Maxima and Minima



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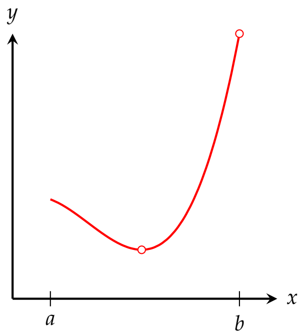
Where does f have a local maximum?

Where does f have a local minimum?

Moral: To find local maxima and minima:

- Find the critical points of $f(x)$
- Apply the second derivative test to each critical point

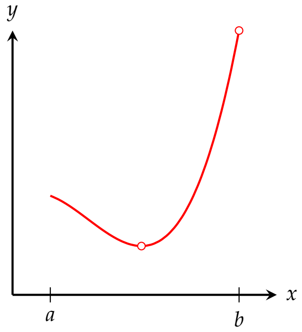
Review - Global Maxima and Minima



A function $f(x)$ has a *global minimum* at $x = c$ if $f(c) \leq f(x)$ for *all* x in the domain of $f(x)$

A function $f(x$ has a *global maximum* at $x = c$ if $(c) \geq f(x)$ for *all* x in the domain of $f(x)$.

Review - Global Maxima and Minima



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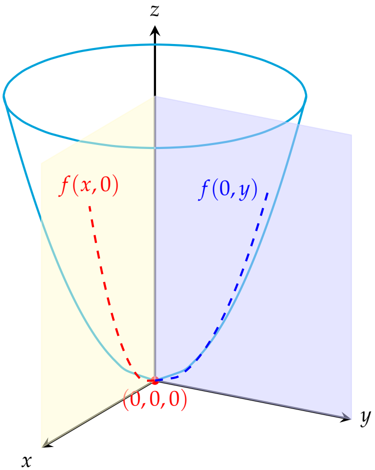
To find the global maximum and minimum of $f(x)$ on $[a, b]$:

- Find the critical points of $f(x)$ on (a, b)
- Evaluate $f(x)$ at the critical points and the endpoints $x = a$ and $x = b$

When does $f(x,y)$ have a critical point?

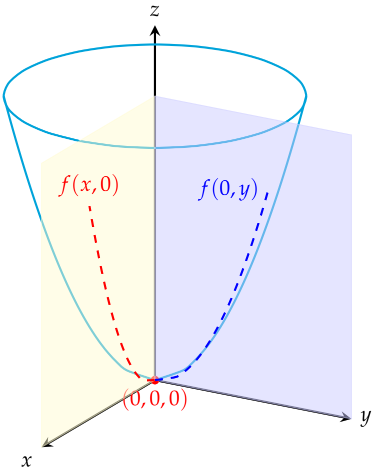


$f(x,y) = x^2 + y^2$, $(a,b) = (0,0)$

If $f(x,y)$ has a local maximum or minimum at (a,b):

- The function $g(x) = f(x,b)$ has a local maximum or minimum
- The function $h(x) = f(a,y)$ has a local maximum or minimum

When does $f(x,y)$ have a critical point?



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So:

$$g'(a) = \frac{\partial f}{\partial x}(a,b) = 0 \text{ or DNE}$$

$$h'(b) = \frac{\partial f}{\partial y}(a,b) = 0 \text{ or DNE}$$

Puzzler #1

Find the critical point(s) of the function

$$f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$$

Note that

$$\frac{\partial f}{\partial x} = 2x - 2y + 2, \qquad \frac{\partial f}{\partial y} = -2x + 4y - 6$$

Courtesy of CLP Example 2.9.6

Puzzler #2

Find the critical points of the function

$$f(x,y) = x^3 + y^3 - 3xy$$

Note that

$$\frac{\partial f}{\partial x} = 3x^2 - 3y, \qquad \frac{\partial f}{\partial y} = 3y^2 - 3x$$

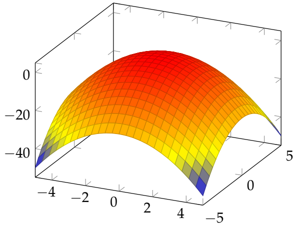
Courtesy of Paul's Online Math Notes

Local Minima

Recall that $f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$ has a single critical point at $(1,2)$. Is it a local maximum, local minimum, or neither?

Compute $f(1+\Delta x, 2+\Delta y)$ and see that happens!

Local Maxima



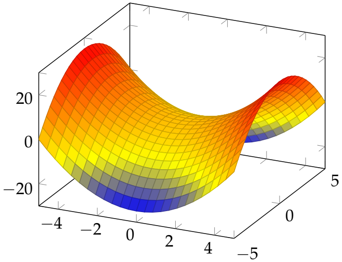
Let

$$f(x, y) = -x^2 - y^2$$

and note that $f(x, y)$ has a single critical point at $(0, 0)$.

$$f(0 + \Delta x, 0 + \Delta y) = -(\Delta x)^2 - (\Delta y)^2$$

Local Neither



Let

$$f(x, y) = x^2 - y^2.$$

Note that $f(x, y)$ has a critical point at $(0, 0)$.

But

$$f(0 + \Delta x, 0 + \Delta y) = (\Delta x)^2 - (\Delta y)^2$$

Looking Under the Hood

Remember quadratic approximation?

$$f(a+\Delta x, b+\Delta y) = f(a,b) + \left(\frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y\right)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)\Delta x \Delta y + \frac{\partial^2 f}{\partial y}^2(a,b)(\Delta y)^2\right)$$

If (a,b) is a critical point, then

$$f(a+\Delta x, b+\Delta y) = f(a,b)$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(a,b)(\Delta y)^2\right)$$

so, the second derivatives of f determine whether (a,b) is a local maximum, a local minimum, or neither.

The Second Derivative Test

Suppose that $f(x,y)$ has a critical point at (a,b) and that second derivatives of $f(x,y)$ exist and are continuous near (a,b). Let

$$D(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

- If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$, then $f(x,y)$ has a local minimum at (a,b)
- If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$, then $f(x,y)$ has a local maximum at (a,b)
- If $D(a,b) < 0$ then $f(x,y)$ has neither a local maximum nor a local minimum near (a,b)
- If $D(a,b) = 0$ then the second derivative test fails

The Second Derivative Test

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

Minimum if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$

Maximum if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$

Determine whether $(1,2)$ is a local maximum, local minimum, or neither if $f(x,y) = x^2 - 2xy + 2y^2 + 2x - 6y + 12$

The second derivatives are:

$$f_{xx}(1,2) = 2 \qquad f_{xy}(1,2) = -2$$
$$f_{xy}(1,2) = -2 \qquad f_{yy}(1,2) = 4$$

$$D = \begin{vmatrix} 2 & -2 \\ -2 & 4 \end{vmatrix} = 8 - 4 = 4 > 0$$

$$f_{xx}(1,2) = 2 > 0$$

So $f(x,y)$ has a local minimum at $(1,2)$.

Puzzler #3

$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$
Minimum if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$
Maximum if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$

Find and classify the critical points of

$$f(x,y) = 4 + x^3 + y^3 - 3xy$$

The first and second partials are:

$$f_x(x,y) = 3x^2 - 3y \qquad f_y(x,y) = 3y^2 - 3x$$
$$f_{xx}(x,y) = 6x \qquad f_{yy}(x,y) = 6y$$
$$f_{xy}(x,y) = -3$$

Courtesy of Paul's Online Math Notes

Puzzler #3 Solution

We know that the critical points are $(0,0)$ and $(1,1)$ from earlier in the lecture.

For the critical point $(0,0)$:

$$\begin{vmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9$$

so $(0,0)$ is neither a local maximum nor a local minimum (if you look at the graph you'll see that it's a saddle point)

For the critical point $(1,1)$:

$$\begin{vmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{xy}(0,0) & f_{yy}(0,0) \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix}$$

$$= 36 - 9 = 27 > 0$$

and $f_{xx}(1,1) = 6$ so $(1,1)$ is a local minimum of $f(x,y)$.

Reminders for the Week of October 9-13

- Homework B5 due at 11:59 Monday, October 9
- Quiz #6 on linear approximation, directional derivatives, and the gradient due at 11:59 Thursday, October 12
- Homework B6 on maxima and minima due at 11:59 on Friday, October 13