Math 213 - Global Maxima and Minima

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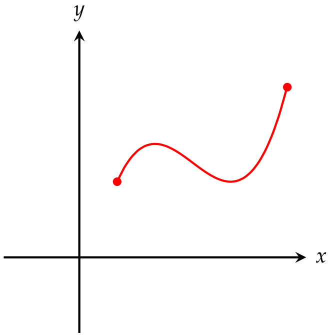
October 9, 2023

Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

Absolute Maxima and Minima, One Variable



Theorem: If $f(x)$ is continous on $[a, b]$ then $f(x)$ has an absolute maximum value and an absolute minimum value on $[a, b]$

To find the absolute maximum and minimum values of $f(x)$ on $[a, b]$:

- Find the critical points of $f(x)$ in (a, b)
- Find the value of $f(x)$ at the critical points and at the endpoints

Example: $f(x) = x^3 - \frac{9}{2}x^2 + 6x - 1$ on $[\frac{1}{2}, \frac{9}{4}]$

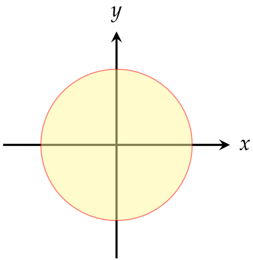
$$f'(x) = 3(x-1)(x-2)$$

Absolute Maxima and Minima, Two Variables

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 - x - y + 1$ on the unit disk $x^2 + y^2 \leq 1$.

Step 1: Find interior critical points

$$f_x(x, y) = 2x - 1$$
$$f_y(x, y) = 2y - 1$$



Solve

$$2x - 1 = 0$$
$$2y - 1 = 0$$

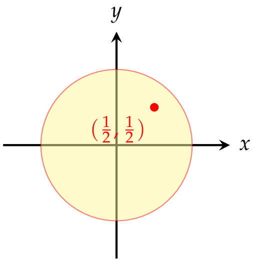
and compute

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$$

Problem courtesy of our colleagues at University of Texas

Absolute Maxima and Minima, Two Variables

Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - x - y + 1$ on the unit disk $x^2 + y^2 \leq 1$.



Step 2: Find the maximum and minimum of $f(x,y)$ on the boundary of the unit disc

Parametrize the circle

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$

for $0 \leq t \leq 2\pi$ and find the maximum and minimum of

$$h(t) = f(x(t), y(t)) = 2 - \cos(t) - \sin(t)$$

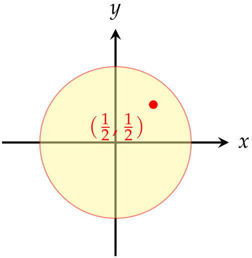
on the interval $[0, 2\pi]$

Absolute Maxima and Minima, Two Variables

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 - x - y + 1$ on the unit disk $x^2 + y^2 \leq 1$.

Step 2 continued:

$$h(t) = 2 - \cos(t) - \sin(t)$$
$$h'(t) = \sin(t) - \cos(t)$$



$h'(t) = 0$ if $t = \pi/4, 5\pi/4$

t	$h(t)$
0	1
$\pi/4$	$2 - \sqrt{2}$
$5\pi/4$	$2 + \sqrt{2}$
2π	1

Value at Critical Point: $\frac{1}{2}$
Maximum on Boundary: $2 + \sqrt{2}$ at $(\sqrt{2}/2, \sqrt{2}/2)$
Minimum on Boundary: $2 - \sqrt{2}$ at $(-\sqrt{2}/2, -\sqrt{2}/2)$

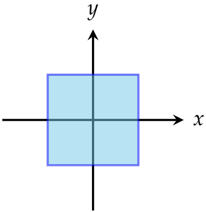
Maximum Value: $2 + \sqrt{2}$ Minimum Value: $2 - \sqrt{2}$

Vocabulary Break

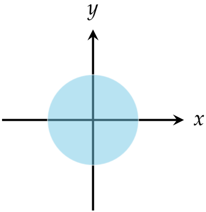
If D is a region in the xy plane:

- D is called *closed* if it contains its boundary points
- D is called *open* if it does not contain any of its boundary points
- D is called *bounded* if it can be contained in a disc of bounded radius

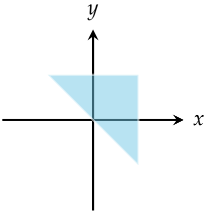
Which of the following regions are open? Which are closed?



$-1 \leq x \leq 1,\ -1 \leq y \leq 1$



$0 \leq x^2 + y^2 < 1$



$x + y > 0$

Absolute Maxima and Minima, Two Variables

Theorem: If $f(x, y)$ is continuous in a closed bounded region D of the xy plane, then f has an absolute maximum value and an absolute minimum value in D.

To find the absolute maximum and minimum values of a function $f(x, y)$ defined on a closed set D in the xy-plane:

1. Find the critical points of $f(x, y)$ in the interior of D
2. Find the maximum and minimum value of $f(x, y)$ on the boundary of D
3. List the values of $f(x, y)$ at the critical points together with its maximum and minimum values on the boundary, and find the largest and smallest values

Puzzler #1

Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ on the region $-1 \leq x \leq 1, -1 \leq y \leq 1$.

Step 1: Find the interior critical points of $f(x,y)$

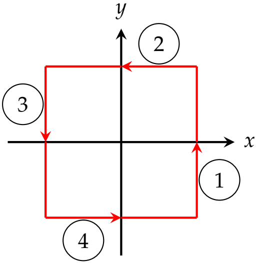
$$f_x(x,y) = 2x - 4xy, \quad f_y(x,y) = 8y - 2x^2$$

Example courtesy of Paul's Online Math Notes

Puzzler #1

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 4y^2 - 2x^2y + 4$ on the region $-1 \leq x \leq 1, -1 \leq y \leq 1$.

Step 2: Find the maximum and minimum value of $f(x, y)$ on the boundary



$$
\begin{array}{llll}
(1) & f(1,y) = 4y^2 - 2y + 5 & -1 \leq y \leq 1 \\
(2) & f(x,1) = -x^2 + 8 & -1 \leq x \leq 1 \\
(3) & f(-1,y) = 4y^2 - 2y + 5 & -1 \leq y \leq 1 \\
(4) & f(x,-1) = 3x^2 + 8 & -1 \leq x \leq 1
\end{array}
$$

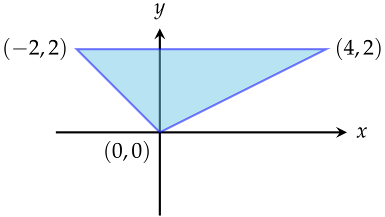
Example courtesy of Paul's Online Math Notes

Puzzler # 2

Find the absolute maximum and minimum values of the function

$$f(x,y) = 192x^3 + y^2 - 4xy^2$$

on the triangle with vertices $(0,0)$, $(4,2)$, and $(-2,2)$



Step 1: Find the interior critical points of $f(x,y)$

$$f_x(x,y) = 576x^2 - 4y^2$$
$$f_y(x,y) = 2y - 8xy$$

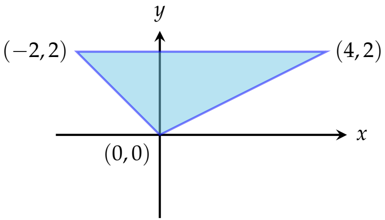
Courtesy of Paul's Online Math Notes

Puzzler # 2

Find the absolute maximum and minimum values of the function

$$f(x, y) = 192x^3 + y^2 - 4xy^2$$

on the triangle with vertices $(0, 0)$, $(4, 2)$, and $(-2, 2)$



Step 2: Find the maximum and minimum of $f(x, y)$ along the three boundaries

$$y = \frac{x}{2}$$
$$y = 2$$
$$y = -x$$

Courtesy of Paul's Online Math Notes

Reminders for the Week of October 9-13

- Homework B5 due at 11:59 Monday, October 9
- Quiz #6 on linear approximation, directional derivatives, and the gradient due at 11:59 Thursday, October 12
- Homework B6 on maxima and minima due at 11:59 on Friday, October 13
- Exam 2 takes place on Wednesday, October 18, 5:00-7:00 PM.
- If you need an alternate exam, you need to use the online form to apply *no later than Friday, October 13 at 5:00 PM*