Math 213 - Lagrange Multipliers

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Unit B: Differential Calculus (and Some Integral Calculus)

- September 18 - Functions of Several Variables
- September 22 - Partial Derivatives
- September 25 - Higher-Order Derivatives
- September 27 - The Chain Rule
- September 29 - Tangent Planes and Normal Lines
- October 2 - Linear Approximation and Error
- October 4 - Directional Derivatives and the Gradient
- October 6 - Maximum and Minimum Values, I
- October 9 - Maximum and Minimum Values, II
- October 11- Lagrange Multipliers
- October 13 -Double Integrals
- October 16 - Double Integrals in Polar Coordinates

Constrained Optimization

A *constrained optimization problem* takes the form

> *Find the maximum (or minimum) value of $f(x, y)$ for (x, y) on the curve $g(x, y) = 0$*

The function $f(x, y)$ is called the *objective function*

The function $g(x, y)$ is called the *constraint function*

Example:

Find the maximum and minimum of the function

$$f(x, y) = x^2 - 10x - y^2$$

on the ellipse whose equation is

$$x^2 + 4y^2 = 16$$

Lagrange Multipliers

Theorem Suppose that $f(x, y)$ and $g(x, y)$ have continuous partial derivatives in a region of the xy plane that contains the surface S given by $g(x, y) = 0$.

Suppose that $\nabla g(x, y) \neq \langle 0, 0 \rangle$ on S.

If f, restricted to the surface S, has a local extreme value at (a, b), there is a number λ so that

$$\frac{\partial f}{\partial x}(a, b) = \lambda \frac{\partial g}{\partial x}(a, b) \tag{1}$$

$$\frac{\partial f}{\partial y}(a, b) = \lambda \frac{\partial g}{\partial x}(a, b) \tag{2}$$

f is the objective function

g is the constraint function

λ is the Lagrange multiplier

Equations (1) and (2) are the *Lagrange equations*

First Example

$$\frac{\partial f}{\partial x}(a,b) = \lambda \frac{\partial g}{\partial x}(a,b)$$
$$\frac{\partial f}{\partial y}(a,b) = \lambda \frac{\partial g}{\partial x}(a,b)$$
$$g(x,y) = 0$$

Find the maximum and minimum of the function

$$f(x,y) = x^2 - 10x - y^2$$

on the ellipse whose equation is

$$x^2 + 4y^2 = 1$$

Objective function: $f(x,y) = x^2 - 10x - y^2$

Constraint function: $g(x,y) = x^2 + 4y^2 - 1$

Lagrange Equations and constraint:

$$\begin{aligned} 2x - 10 &= \lambda(2x) \\ -2y &= \lambda(8y) \\ x^2 + 4y^2 &= 16 \end{aligned}$$

Now what?

First Example, Continued

First rule of Lagrange Method: Eliminate the parameter!

$$f(x,y) = x^2 - 10x - y^2$$

Lagrange Equations:

$$\begin{aligned} 2x - 10 &= 2\lambda x \\ -2y &= 8\lambda y \end{aligned}$$

Constraint Equation:

$$x^2 + 4y^2 = 16$$

First Example, Continued

$$f(x,y) = x^2 - 10x - y^2$$

Lagrange Equations:

$$2x - 10 = 2\lambda x$$
$$-2y = 8\lambda y$$

Constraint Equation:

$$x^2 + 4y^2 = 16$$

First rule of Lagrange Method: Eliminate the parameter!

From the second Lagrange equation, either

1. $y = 0$, or
2. $\lambda = -1/4$

In case (1), we get $x = \pm 4$ from the constraint equation

In case (2), we get $x = 4$ so $y = 0$

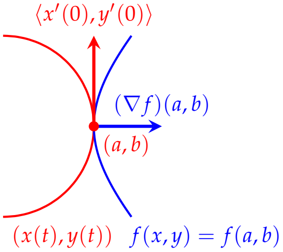
Now test the points $(-4,0)$ and $(4,0)$:

$f(-4,0) = -24$, $f(4,0) = 56$

Conclusion:

Minimum value: -24 Maximum value: 56

Why Does It Work?



Suppose that (a,b) is a local extremum of $f(x,y)$ along $g(x,y) = 0$.

Let $(x(t), y(t))$ be a curve along $g(x,y) = 0$ with $x(0) = a$, and $y(0) = b$.

Then

$$F(t) = f(x(t), y(t))$$

has a local extremum at $t = 0$.

So $F'(0) = 0$ or

$$\frac{\partial f}{\partial x}(a,b) \cdot x'(0) + \frac{\partial f}{\partial y}(a,b) \cdot y'(0) = 0$$

That is, $(\nabla f)(a,b)$ is perpendicular to the tangent vector along any curve on the surface $g(x,y) = 0$

Since $(\nabla g)(a,b)$ is also perpendicular to any vector tangent to the surface,

$$(\nabla f)(a,b) = \lambda(\nabla g)(a,b)$$

Puzzler # 1

$$\frac{\partial f}{\partial x}(a,b) = \lambda \frac{\partial g}{\partial x}(a,b)$$

$$\frac{\partial f}{\partial y}(a,b) = \lambda \frac{\partial g}{\partial x}(a,b)$$

$$g(x,y) = 0$$

Find the maximum and minimum values of $f(x,y) = 3x - 4y$ subject to the constraint $x^2 + y^2 = 25$

Absolute Maxima and Minima

Find the absolute maximum and minimum of $f(x,y) = 4x^2 + 10y^2$ in the region $x^2 + y^2 \leq 4$. Use the Lagrange multiplier method to find the maximum and minimum of $f(x,y)$ on the boundary.

Puzzler # 2

$$\frac{\partial f}{\partial x}(a,b,c) = \lambda \frac{\partial g}{\partial x}(a,b,c)$$
$$\frac{\partial f}{\partial y}(a,b,c) = \lambda \frac{\partial g}{\partial y}(a,b,c)$$
$$\frac{\partial f}{\partial z}(a,b,c) = \lambda \frac{\partial g}{\partial z}(a,b,c)$$
$$g(a,b,c) = 0$$

Find the maximum and minimum values of $f(x,y,z) = xyz$ subject to the constraint $x + y + z = 1$. Assume that $x, y, z \geq 0$.

Problem courtesy of Paul's Online Math Notes

Two Constraints

The method of Lagrange multipliers also works when there are two constraints.

Problem: Find the maximum and minimum values of $f(x,y,z)$ subject to the constraints

$$g(x,y,z) = 0$$
$$h(x,y,z) = 0$$

The new Lagrange equations are

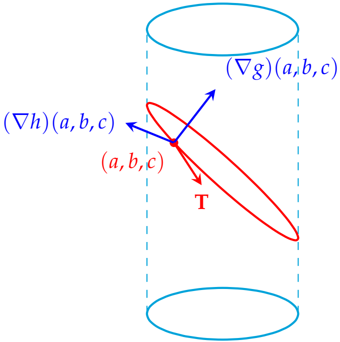
$$(\nabla f)(a,b,c) = \lambda(\nabla g)(a,b,c) + \mu(\nabla h)(a,b,c)$$

with the constraints given above.

Why Does it Work?

Find the maximum and minimum values of $f(x,y,z) = 4y - 2z$ subject to the constraints

$$g(x,y,z) = \underbrace{2x - y - z - 2 = 0}_{\text{plane}} \qquad h(x,y,z) = \underbrace{x^2 + y^2 - 1 = 0}_{\text{cylinder}}$$



If f has a maximum or minimum at (a,b,c), and if $(x(t),y(t),z(t))$ moves along the red curve and passes through (a,b,c) at time $t = 0$, then

$$h(t) = f(x(t),y(t),z(t))$$

satisfes $h'(0) = 0$ or

$$(\nabla f)(a,b,c) \cdot \langle x'(0), y'(0), z'(0) \rangle = 0$$

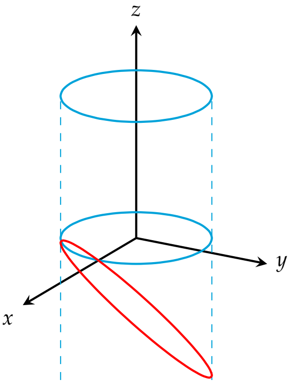
Since the gradient of f is perpendicular to any tangent vector along the curve,

$$(\nabla f)(a,b,c) = \lambda(\nabla g)(a,b,c) + (\nabla h)(a,b,c)$$

Two Constraints

Find the maximum and minimum of $f(x,y,z) = 4y - 2z$ subject to the constraints

$$2x - y - z = 2, \quad x^2 + y^2 = 1$$



$$g(x,y,z) = 2x - y - z - 2,$$
$$h(x,y,z) = x^2 + y^2 - 1$$

$$(\nabla f) = \lambda(\nabla g) + \mu(\nabla h)$$

What are the Lagrange equations and the constraint equations?

Two Constraints

$$f(x,y,z) = 4y - 2z$$
$$2x - y - z = 2$$
$$x^2 + y^2 = 1$$
$$\langle 0,4,-2\rangle = \lambda\langle 2,-1,-1\rangle + \mu\langle 2x,2y,0\rangle$$

Solve:

$$0 = 2\lambda + 2\mu x$$
$$4 = -\lambda + 2\mu y$$
$$-2 = -\lambda$$

or

$$2\mu x = -2\lambda$$
$$2\mu y = -\lambda - 4$$
$$\lambda = 2$$

Two Constraints

$$f(x,y,z) = 4y - 2z$$
$$2x - y - z = 2$$
$$x^2 + y^2 = 1$$
$$\langle 0, 4, -2 \rangle = \lambda \langle 2, -1, -1 \rangle + \mu \langle 2x, 2y, 0 \rangle$$

Solve:

$$0 = 2\lambda + 2\mu x$$
$$4 = -\lambda + 2\mu y$$
$$-2 = -\lambda$$

or

$$2\mu x = -2\lambda$$
$$2\mu y = -\lambda - 4$$
$$\lambda = 2$$

Using $\lambda = 2$:

$$2\mu x = -4$$
$$2\mu y = 6$$

Reminders for the Week of October 9-13

- Quiz #6 on linear approximation, directional derivatives, and the gradient due at 11:59 Thursday, October 12
- Homework B6 on maxima and minima due at 11:59 on Friday, October 13
- Exam 2 takes place on Wednesday, October 18, 5:00-7:00 PM.
- If you need an alternate exam, you need to use the online form to apply *no later than Friday, October 13 at 5:00 PM*