Math 213 - Double Integrals

Peter Perry

October 13, 2023

Exam II Reminders

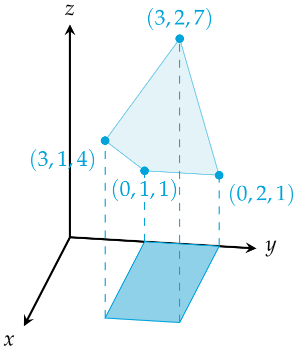
- Your Exam covers webworks B1-B6
- Your Exam takes place on Wednesday October 18, 5:00-7:00 PM
- If you need an alternate exam you need to fill out the request form here no later than 5 PM today
- You are allowed a one-page sheet of notes (both sides OK)
- Section 011 will take the exam in CP 139
- Sections 012-014 will take the exam in CP 153
- You are only in section 011 if you have your recitation with Mr. Berggren from 12:00-12:50 PM TR
- You are in section 012 if you have Mr. Berggren 1:00-1:50 PM TR
- You are in section 013 if you have Mr. Alsteri 2:00-2:50 PM TR
- You are in section 014 if you have Mr. Alsetri 3:00-350 PM TR

Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 18 - Exam II Review
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

Introduction to Double Integrals

Problem: Find the volume that lies between the rectangle $0 \le x \le 3$, $1 \le y \le 2$ and the graph of $f(x, y) = 1 + xy$



We can approximate the volume under the graph of $f(x, y)$ in the following way:

- Divide the rectangle into subrectangles
- Over each rectangle, make a box that touches the graph of f on top
- Add up the volumes of all the boxes

Introduction to Double Integrals

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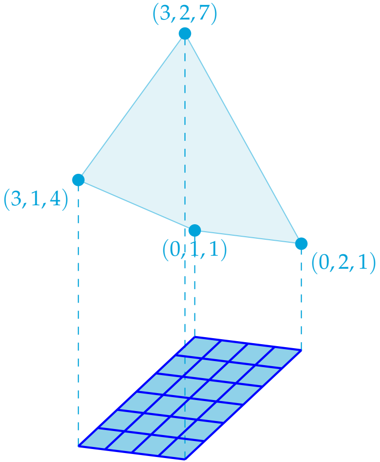

- Divide the rectangle into subrectangles of area $\Delta x \cdot \Delta y$ with center at (x_i, y_j)

Introduction to Double Integrals

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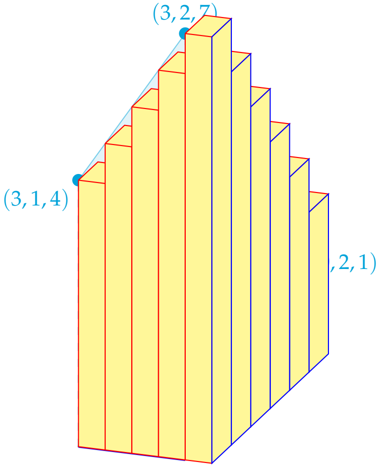

- Divide the rectangle into subrectangles of area $\Delta x \cdot \Delta y$ with center at (x_i, y_j)
- Over each rectangle, make a box of height $f(x_i, y_j)$

Introduction to Double Integrals

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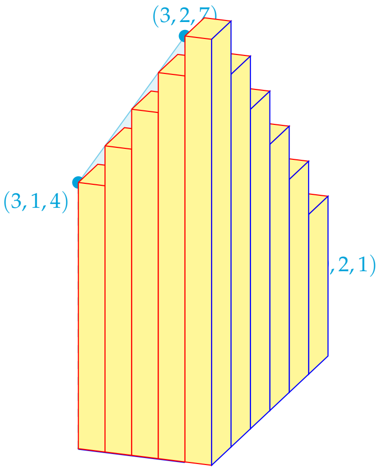

- Divide the rectangle into subrectangles of area $\Delta x \cdot \Delta y$ with center at (x_i, y_j)
- Over each rectangle, make a box of height $f(x_i, y_j)$
- Approximate the volume by

$$V \simeq \sum_{i,j} f(x_i, y_j) \Delta x \Delta y$$

If this looks like a Riemann sum to you, it is!

Computing the Double Integral

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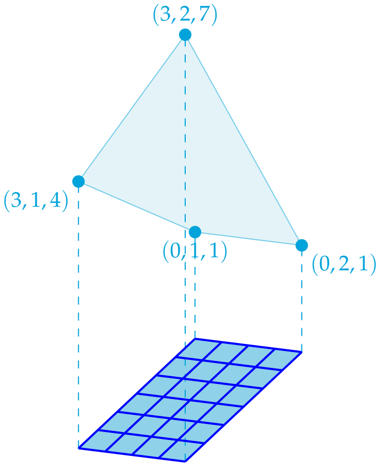


$$V \simeq \sum_{i,j} f(x_i, y_j)\Delta x \Delta y$$

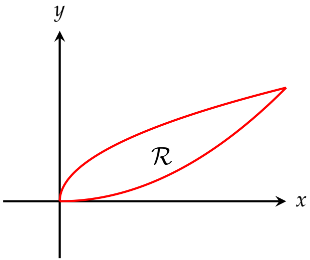
$$0 \leq x \leq 3, \quad 1 \leq y \leq 2$$

The Riemann sum converges to the *double integral*

$$V = \int_0^3 \int_1^2 (1 + xy)\, dy\, dx$$

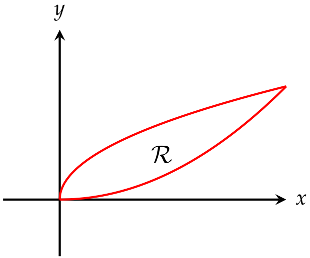
which can be evaluated "from the inside out"

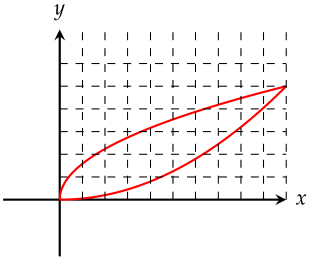
What if the Region isn't a Rectangle?



A region $\mathcal{R}$ bounded by the curves $4y = x^2$ and $x = 2y^2$ has density xy. What is the total mass of the region?

What if the Region isn't a Rectangle?





A region $\mathcal{R}$ bounded by the curves $4y = x^2$ and $x = 2y^2$ has density xy. What is the total mass of the region?

New idea: divide up the region into rectangles B_{ij} of width Δx and height Δy with center (x_i^*, y_j^*)

The mass of each square is

$$M_{ij} = (x_i y_j)\,\Delta x \Delta y$$

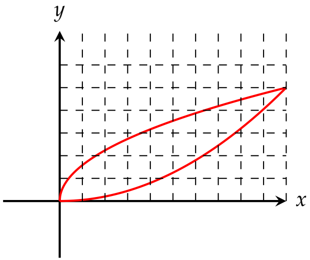
The total mass is approximately

$$M \simeq \sum_{i,j} M_{ij} = \sum_{i,j} (x_i y_j)\,\Delta x\,\Delta y$$

and exactly the *double integral*

$$M = \iint_{\mathcal{R}} xy\,dA$$

What if the Region isn't a Rectangle?



Top curve: $x = 2y^2$

Bottom curve: $4y = x^2$

Region:

$$0 \leq x \leq 2, \quad x^2/4 \leq y \leq \sqrt{x/2}$$

$$M = \iint_{\mathcal{R}} xy \, dA$$

$$= \int_0^2 \underbrace{\left(\int_{x^2/4}^{\sqrt{x/2}} xy \, dy \right)}_{\text{a function of } x} dx$$

Evaluating a Double Integral - Inside Out

$$\iint_{\mathcal{R}} xy\,dA = \int_0^2 \underbrace{\left(\int_{x^2/4}^{\sqrt{x/2}} xy\,dy \right)}_{\text{a function of } x} dx$$

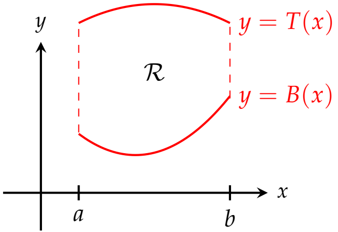
$$= \int_0^2 \left(\frac{xy^2}{2} \Bigg|_{y=x^2/4}^{y=\sqrt{x/2}} \right) dx$$

$$= \int_0^2 \left(\frac{x^2}{4} - \frac{x^5}{32} \right) dx$$

$$= \left[\frac{x^3}{12} - \frac{x^6}{188} \right] \Bigg|_{x=0}^{x=2}$$

$$= \frac{8}{12} - \frac{64}{188}$$

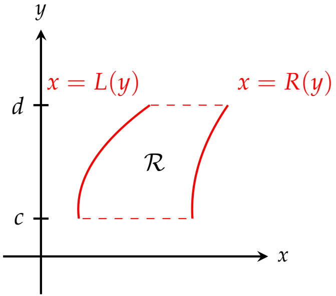
Double Integrals are Iterated Integrals



$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_a^b \underbrace{\left(\int_{B(x)}^{T(x)} f(x,y)\,dy\right)}_{\text{a function of } x} dx$$

Puzzler # 1

Find an iterated integral for $\iint_{\mathcal{R}} f(x,y)\,dA$ is $\mathcal{R}$ is the region shown below.



For a "left to right" region,

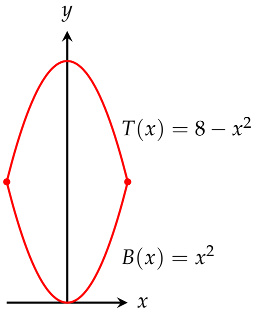
$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_c^d \left(\int_{L(y)}^{R(y)} f(x,y)\,dx \right) dy$$

Puzzler # 2

Find the volume that lies below the surface $z = 16xy + 200$ and above the region in the xy plane bounded by $y = x^2$ and $y = 8 - x^2$

Basic idea: If $h(x, y)$ is the height of the surface at (x, y), the volume over a region $\mathcal{R}$ should be

$$V = \iint_{\mathcal{R}} h(x, y)\, dA$$



Answer: The region is

$$-2 \le x \le 2, \quad x^2 \le y \le 8 - x^2.$$

So

$$\iint_{\mathcal{R}} (16xy + 100)\, dA = \int_{-2}^{2} \int_{x^2}^{8-x^2} (16xy + 100)\, dy\, dx$$

Properties of Double Integrals

Additive Properties:

$$\int_{\mathcal{R}} (f(x,y) + g(x,y))\, dA = \int_{\mathcal{R}} f(x,y)\, dA + \int_{\mathcal{R}} g(x,y)\, dA$$

$$\int_{\mathcal{R}} C f(x,y)\, dA = C \int_{\mathcal{R}} f(x,y)\, dA$$

If a region $\mathcal{R}$ in the xy plane is the union of regions $\mathcal{R}_1$ and $\mathcal{R}_2$, then

$$\int_{\mathcal{R}} f(x,y)\, dA = \int_{\mathcal{R}_1} f(x,y)\, dA + \int_{\mathcal{R}_2} f(x,y)\, dA$$

Notation: The following notations are all used for the double integral:

$$\int_{\mathcal{R}} f(x,y)\, dA = \int_{\mathcal{R}} f(x,y)\, dx\, dy = \int_{\mathcal{R}} f(x,y)\, dy\, dx$$

all mean the same thing.

Properties of Double Integrals

Note that $\int_{\mathcal{R}} dA = \text{area}(\mathcal{R})$

Inequalities:

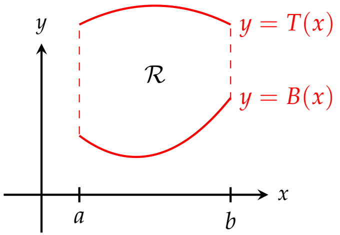
- If $f(x,y) \geq 0$, then $\int_{\mathcal{R}} f(x,y)\,dA \geq 0$
- If $f(x,y) \geq g(x,y)$, then $\int_{\mathcal{R}} f(x,y)\,dA \geq \int_{\mathcal{R}} g(x,y)\,dA$
- If $m \leq f(x,y) \leq M$, then

$$m\,\text{area}(\mathcal{R}) \leq \int_{\mathcal{R}} f(x,y)\,dA \leq M\,\text{area}(\mathcal{R})$$

Swapping Variables

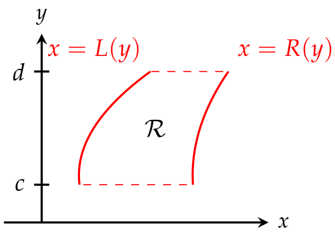
Sometimes, a region of integration can be described either by the "bottom to top" method or the "left to right" method, and one method may be much easier to use than the other!

"Top-Bottom" Method



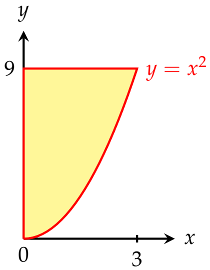
$$\int_{\mathcal{R}} f(x,y)\,dA = \int_a^b \left(\int_{B(x)}^{T(x)} f(x,y)\,dy \right) dx$$

"Left-Right Method"



$$\int_{\mathcal{R}} f(x,y)\,dA = \int_c^d \left(\int_{L(y)}^{R(y)} f(x,y)\,dx \right) dy$$

Swapping Variables

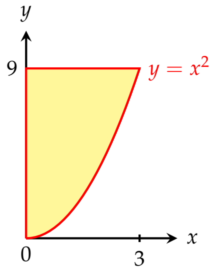


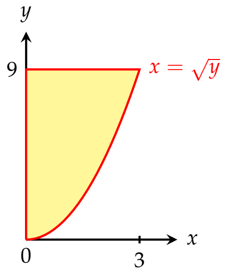
Find the iterated integral

$$\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\, dy\, dx$$

by changing the order of integration

Swapping Variables





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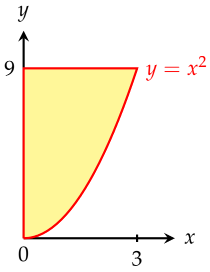

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Answer: rewrite the integral as

$$\int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3}\, dx\, dy = \int_0^9 \left[\frac{x^4}{4} e^{y^3} \right]_0^{\sqrt{y}} dy$$

$$= \int_0^9 \frac{y^2}{4} e^{y^3}\, dy$$

$$= \frac{1}{12} \int_0^{729} e^u\, du$$

$$= \frac{e^{729}}{12}$$

Reminders for the Week of October 16-20

- Homework B6 is due tonight at 11:59 PM
- Double Integrals are covered in Recitaton on Tuesday October 17
- Your second exam takes place on Wednesday, October 18, 5:00-7:00 PM
- There is no recitation Thursday, October 19