Math 213 - Double Integrals in Polar Coordinates

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October 16, 2023

Exam II Reminders

- Your Exam covers webworks B1-B6
- Your Exam takes place on Wednesday October 18, 5:00-7:00 PM
- You are allowed a one-page sheet of notes (both sides OK)
- Section 011 will take the exam in CP 139
- Sections 012-014 will take the exam in CP 153
- You are only in section 011 if you have your recitation with Mr. Berggren from 12:00-12:50 PM TR
- You are in section 012 if you have Mr. Berggren 1:00-1:50 PM TR
- You are in section 013 if you have Mr. Alsteri 2:00-2:50 PM TR
- You are in section 014 if you have Mr. Alsetri 3:00-350 PM TR

Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 18 - Exam II Review
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

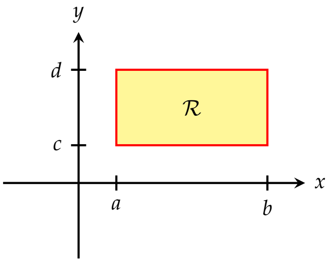
Review

We learned to compute the double integral

$$\iint_{\mathcal{R}} f(x,y)\, dA$$

of a function $f(x,y)$ over a region $\mathcal{R}$ in the xy plane:

(1) Over a rectangle:



$$\iint_{\mathcal{R}} f(x,y)\, dA = \int_a^b \underbrace{\left(\int_c^d f(x,y)\, dy\right)}_{\text{function of } x} dx$$

or also

$$\iint_{\mathcal{R}} f(x,y)\, dA = \int_c^d \underbrace{\left(\int_a^b f(x,y)\, dx\right)}_{\text{function of } y} dy$$

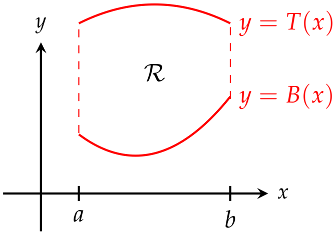
Review

We learned to compute the double integral

$$\iint_{\mathcal{R}} f(x,y)\,dA$$

of a function $f(x,y)$ over a region $\mathcal{R}$ in the xy plane:

(2) Over a "bottom-top" region:



$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_a^b \underbrace{\left(\int_{B(x)}^{T(x)} f(x,y)\,dy \right)}_{\text{function of } x} dx$$

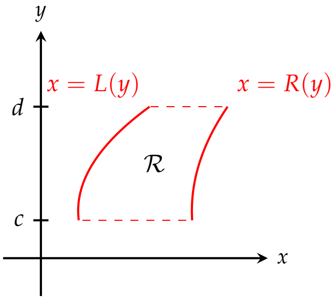
Review

We learned to compute the double integral

$$\iint_{\mathcal{R}} f(x,y)\,dA$$

of a function $f(x,y)$ over a region $\mathcal{R}$ in the xy plane:

(3) Over a "left-right" region:



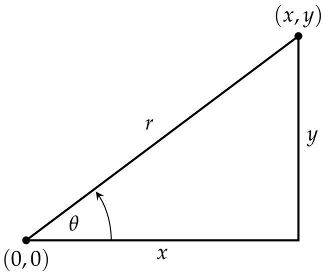
$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_c^d \underbrace{\left(\int_{L(y)}^{R(y)} f(x,y)\,dx\right)}_{\text{function of } y} dy$$

Double Integrals in Polar Coordinates

Today we'll learn how to compute double integrals $\iint_{\mathcal{R}} f(x,y)\,dA$ when the region $\mathcal{R}$ is given in *polar coordinates*

Polar Coordinate Review

You can remember the conversions to and from polar coordinates by the triangle below



From Cartesian to polar coordinates:

$$r = \sqrt{x^2 + y^2}$$
$$\tan\theta = \frac{y}{x}$$

From polar to Cartesian coordinates:

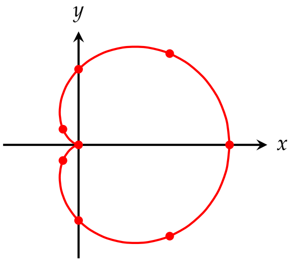
$$x = r\cos\theta$$
$$y = r\sin\theta$$

Polar Curves

The boundaries of polar regions are sometimes given by *polar curves*.

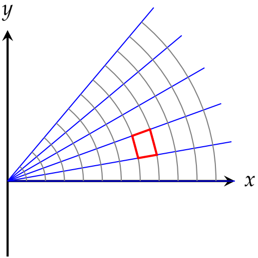
Example: the Cardioid

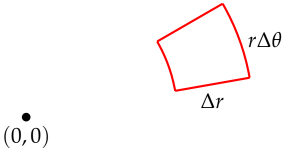
$$r = 1 + \cos(\theta)$$



θ	$1+\cos(\theta)$
0	2
$\pi/4$	$1+\sqrt{2}/2$
$\pi/2$	1
$3\pi/4$	$1-\sqrt{2}/2$
π	0
$5\pi/4$	$1-\sqrt{2}/2$
$3\pi/2$	1
$7\pi/4$	$1+\sqrt{2}/2$
2π	2

Integrals in Polar Coordinates





To set up an integral in polar coordinates:

- Set up a polar coordinate grid with "polar rectangles" of side Δr and $r\Delta\theta$
- Find that the area of a small "polar rectangle" is $\Delta A = r\Delta r\Delta\theta$
- Guess that

$$\sum_{i,j} f(r_i \cos(\theta_j), r_i \sin(\theta_j))\Delta A$$

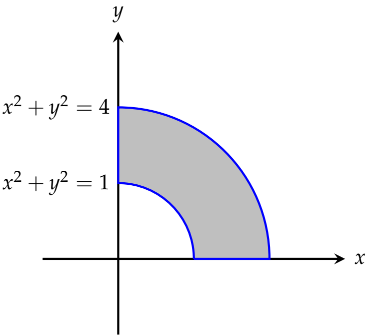
becomes

$$\iint f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Test Case: Polar Rectangles

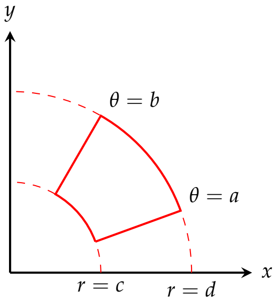
$dA = r\,dr\,d\theta$

Find a polar integral for $\iint_{\mathbf{R}} f(x,y)\,dA$ if **R** is the region shown at left below.



Polar Rectangles

The region $a \leq \theta \leq b$ and $c \leq r \leq d$ is called a *polar rectangle*



If $\mathcal{R}$ is a polar rectangle,

$$\iint_{\mathcal{R}} f(x,y)\, dA = \int_a^b \int_c^d f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

Puzzler #1

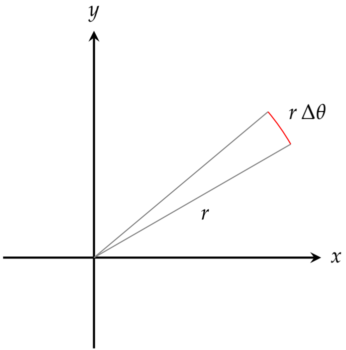
If $a \leq \theta \leq b$ and $c \leq r \leq d$ then

$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_a^b \int_c^d f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Find $\iint_{\mathcal{R}} 2xy\,dA$ if $\mathcal{R}$ is the region between the circles of radius 2 and 5 in the first quadrant.

Example Courtesy of Paul's Online Math Notes

Area Enclosed by a Polar Curve



$$\Delta A = \frac{1}{2} r^2 \Delta\theta$$

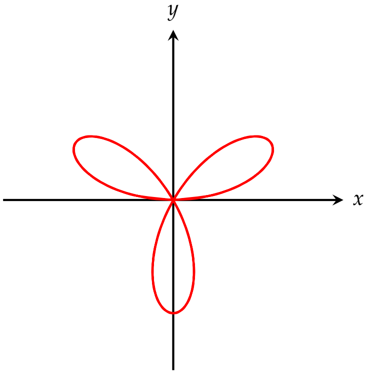
The area enclosed by a polar curve with $a \leq \theta \leq b$ is

$$A = \frac{1}{2} \int_a^b r(\theta)^2 \, d\theta$$

Puzzler #2

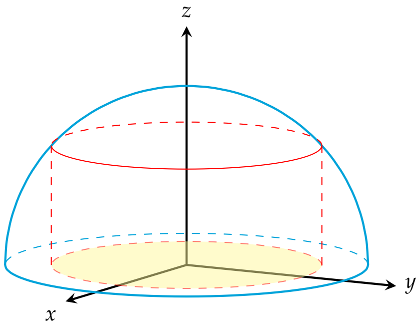
$$A = \frac{1}{2}\int_a^b r(\theta)^2\,d\theta$$

Find the area of one petal of the three-leafed rose $r = \sin(3\theta)$.



θ	$\sin(3\theta)$
0	0
$\pi/6$	1
$\pi/3$	0
$\pi/2$	-1
$2\pi/3$	0
$5\pi/6$	-1
π	0

Volume in Polar Coordinates



Find the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$, and inside the cylinder $x^2 + y^2 = 5$.

Example courtesy of Paul's Online Math Notes

Polar Coordinates Make Hard Problems Easy

Find $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$ by computing I^2 in polar coordinates.

Example courtesy of William Thomson, Lord Kelvin

Reminders for the Week of October 16-20

- Double Integrals are covered in Recitation on Tuesday October 17
- Your second exam takes place on Wednesday, October 18, 5:00-7:00 PM
- There is no recitation Thursday, October 19