# Math 213 - Exam II Review

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# Exam II Reminders

- Your Exam covers webworks B1-B6
- Your Exam takes place tonight, 5:00-7:00 PM
- You are allowed a one-page sheet of notes (both sides OK)
- Section 011 will take the exam in CP 139
- Sections 012-014 will take the exam in CP 153
- You are only in section 011 if you have your recitation with Mr. Berggren from 12:00-12:50 PM TR
- You are in section 012 if you have Mr. Berggren 1:00-1:50 PM TR
- You are in section 013 if you have Mr. Alsteri 2:00-2:50 PM TR
- You are in section 014 if you have Mr. Alsetri 3:00-350 PM TR

# Overview

Exam II will cover the following topics:

- Functions of several variables, limits
- Partial Differentiation basics, including the chain rule
- Tangent planes and normal lines
- Linear approximation and error
- Directional derivatives and the gradient
- Local maxima and minima of functions of two variables (second derivative test)
- Absolute Maxima and Minima on a Bounded Closed Region
- Lagrange multiplier method for constrained optimization problems

# Limits

Find $\lim_{(x,y)\to(0,0)} \dfrac{2x^2 - xy - y^2}{x^2 - y^2}$ or show that it does not exist.

If $f(x,y) = \frac{2x^2-xy-y^2}{x^2-y^2}$, observe that

$$f(x,0) = 2$$
$$f(0,y) = 1$$

so the limits as $x \to 0$ along the line $y = 0$ and as $y \to 0$ along the line $x = 0$ do not agree. The limit does not exist.

Find $\lim_{(x,y)\to(0,0)} \dfrac{x^3 y}{x^6 + y^2}$ or show that it does not exist.

If $f(x,y) = \frac{x^3y}{x^6+y^2}$ then

$$f(x,0) = 0$$
$$f(x,x^3) = \frac{x^6}{2x^6} = \frac{1}{2}$$

so the limits as $x \to 0$ along the lines $y = 0$ and $y = x^3$ do not agree. The limit does not exist.

# Partial Derivatives

Find the first and second partial derivatives of

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

$f_x(x,y) =$

$f_y(x,y) =$

$f_{xx}(x,y) =$

$f_{xy}(x,y) =$

$f_{yx}(x,y) =$

$f_{yy}(x,y) =$

See answers later on!

# Tangent Planes and Normal Lines

Find the equation of the tangent plane and normal line to the surface $z = x^2 + y^2$ at the point $(1, 0, 1)$.

If $f(x, y) = x^2 + y^2$ the surface is the graph of $f$. The normal at $(1, 0, 1)$ is given by $\langle -\frac{\partial f}{\partial x}(1, 0), -\frac{\partial f}{\partial y}(1, 0), 1\rangle = \langle -2, 0, 1\rangle$
The equation of the tangent plane is

$$-2(x - 1) + (z - 1) = 0$$

and the equation of the normal line is

$$\mathbf{r}(t) = \langle 1, 0, 1\rangle + t\langle -2, 0, 1\rangle$$

Problem courtesy of CLP-3 Section 2.5

# Tangent Planes and Normal Lines

Find the tangent plane and normal line to the graph of $x^2 + y^2 + z^2 = 30$ at the point $(1, -2, 5)$

The normal to the surface is $(\nabla g)(1, -2, 5)$ where $g(x, y, z) = x^2 + y^2 + z^2$. Since $(\nabla g)(x, y, z) = \langle 2x, 2y, 2z \rangle$, the normal is $\langle 2, -4, 10 \rangle$.
The equation of the tangent plane is

$$2(x - 1) - 4(y + 2) + 10(z = 5) = 0$$

and the equation of the normal line is

$$\mathbf{r}(t) = \langle 1, -2, 5 \rangle + t\langle 2, -4, 10 \rangle.$$

Problem courtesy of Paul's Online Math Notes

# Linear Approximation

Find an approximate value for $\dfrac{(0.998)^3}{1.003}$

First, find the linear approximation to $f(x,y) = x^3/y$ at $(1,1)$. We have $f(1,1) = 1$ and

$$\frac{\partial f}{\partial x} = 3x^2/y, \qquad \frac{\partial f}{\partial x}(1,1) = 3$$
$$\frac{\partial f}{\partial y} = -x^3/y^2, \qquad \frac{\partial f}{\partial y}(1,1) = -1$$

So

$$L(x,y) = 1 + 3(x-1) - (y-1)$$

Now evaluate

$$L(0.998, 1.003).$$

Problem courtesy of CLP-3 Section 2.6

# Directional Derivatives, Gradient

If **u** is a unit vector, the directional derivative of $f(x,y)$ at $(a,b)$ in the direction **u** is

$$D_{\mathbf{u}} f(a,b) = (\nabla f)(a,b) \cdot \mathbf{u}$$

The directional derivative of $g(x,y,z)$ at $(a,b,c)$ in the direction **u** is

$$D_{\mathbf{u}} g(a,b,c) = (\nabla g)(a,b,c) \cdot \mathbf{u}$$

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Find the directional derivative of $w(x,y,z) = xyz + \ln(xz)$ at $(1,3,1)$ in the direction of $\langle 1,0,1 \rangle$ Some hints: Choose $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$, a unit vector in the direction of $\langle 1,0,1 \rangle$ and take the dot product of **u** with $(\nabla w)(1,3,1)$.

Problem courtesy of CLP-3 Section 2.7

# More About the Gradient

$(\nabla f)(a,b)$ points in the direction of greatest rate of change of $f$ at $(a,b)$ and its magnitude is the greatest rate of change

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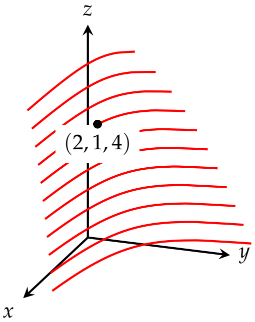


A hiker is walking up a mountain with height

$$h(x,y) = 6 - xy^2$$

where the positive $x$-axis points east and the positive $y$ axis points north. The hiker starts out at $P(2,1,4)$.

In what direction should the hiker proceed from $P$ to ascend on the steepest path?

We found that $(\nabla h)(2,1) = \langle -1, -4 \rangle$ so the hiker should proceed south west.

Problem courtesy of CLP-3 Section 2.7, Problem 8

# Local Maxima and Minima

Find and classify the local maxima and minima of

$$f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$$

Remember that

$$f_x(x,y) = 6x(y-1) \qquad f_y(x,y) = 3x^2 + 3y^2 - 6y$$

$$f_{xx}(x,y) = 6y - 6 \qquad f_{xy}(x,y) = 6x$$
$$f_{yx}(x,y) = 6x \qquad f_{yy}(x,y) = 6y - 6$$

The critical points are $(0,0)$, $(0,2)$, $(1,1)$ and $(-1,1)$. You can apply the second derivative test and find that these points are, in order, a local maximum, a local minimum, and two saddle points.

# Absolute Maxima and Minima

Find the absolute maximum and minimum of the function

$$T(x,y) = (x+y)e^{-(x^2+y^2)}$$

on the region $x^2 + y^2 \leq 1$.

This function has an interior critical point at $(1/2, 1/2)$ with $f(1/2,1/2) = e^{-1/2} \sim 0.606$.

The boundary is parametrized by $x = \cos\theta, y = \sin\theta, 0 \leq \theta \leq 2\pi$. You can work out that

$$f(\cos\theta, \sin\theta) = (\cos\theta + \sin\theta)e^{-1}$$

and there are local extrema at $\theta = \pi/4, \theta = 5\pi/4$. Testing the critical points and endpoints we get

| $\theta$ | $f(\cos\theta, \sin\theta)$ |
|---|---|
| $0$ | $e^{-1} \sim 0.367$ |
| $\pi/4$ | $\sqrt{2}e^{-1} \sim 0.520$ |
| $5\pi/4$ | $-\sqrt{2}e^{-1} \sim -0.520$ |

so the minimum is $-\sqrt{2}e^{-1}$ and the maximum is $e^{\frac{1}{2}}$.

Problem courtesy of CLP-3 Section 2.9, Example 2.9.21

# Lagrange Multiplier Method

Find the maximum and minimum of the function

$$f(x,y) = x^2 - 10x - y^2$$

on the ellipse whose equation is $x^2 + 4y^2 = 16$.

Lagrange equations:

$$2x - 10 = 2\lambda x$$
$$-2y = 8\lambda y$$

or

$$2x(1-\lambda) = 10$$
$$2y(4\lambda + 1) = 0$$

so either $y = 0$ or $\lambda = -1/4$. You can check that this leads to $(x,y) = (\pm 4, 0)$. From $f(4,0) = -24$, $f(-4,0) = 56$ we obtain the maximum and minimum of $f$ on the ellipse.

Problem courtesy of CLP-3 section 2.10, Example 2.10.3

# Reminders for the Week of October 16-20

- Your exam is tonight, 5:00-7:00 PM
- There is no recitation Thursday, October 19
- On Friday, October 20, we will begin discussing triple integrals