THE CORED APPLE PROBLEM

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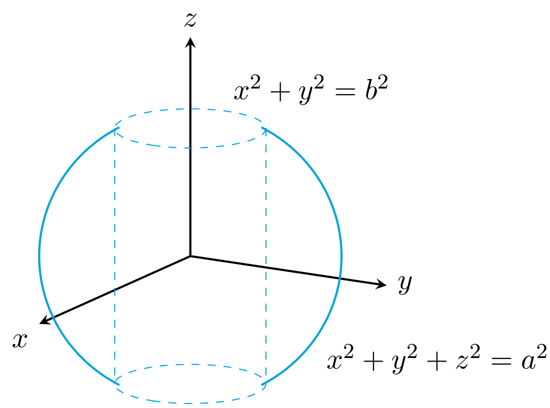
FIGURE 1. The Cored Apple



Problem: A sphere of radius a has a cylindrical hole of radius b drilled out. What is the remaining volume?

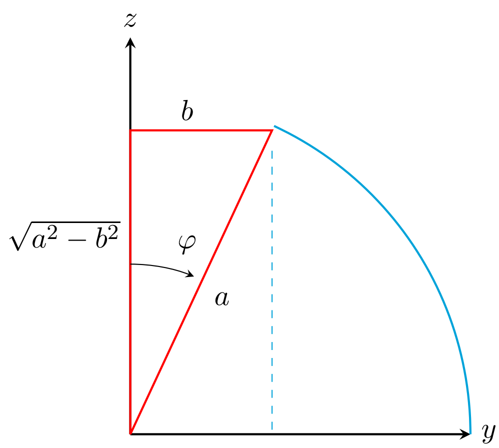
We can compute the volume as

$$V = \iiint_{\mathcal{R}} dV$$

if we can describe the region $\mathcal{R}$ in spherical coordinates. To simplify the problem we'll only consider the top half of the cylinder, and at the end multiply by 2 to answer the original question.

To help with this consider the projection of the solid onto the yz plane:

FIGURE 2. Cross Section of the Cored Apple



From the figure we see that the remaining region begins at

$$\varphi_0 = \tan^{-1}\left(\frac{b}{\sqrt{a^2 - b^2}}\right)$$

Next, we need to figure out, for each φ and θ, the allowed values of ρ, beginning at the cylinder and ending at $\rho = a$.

Substitute $x = \rho \sin\varphi \cos\theta$ and $y = \rho \sin\varphi \sin\theta$ into the equation of the cylinder:

$$\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta = b^2$$

and solve for ρ in terms of φ:

$$\begin{aligned} \rho^2 \sin^2\varphi &= b^2 \\ \rho &= \frac{b}{\sin\varphi} \end{aligned}$$

We now can describe the region $\mathcal{R}$:

$$\begin{aligned} \frac{b}{\sin\varphi} &\leq \rho \leq a \\ 0 &\leq \theta \leq 2\pi \\ \tan^{-1}\frac{b}{\sqrt{a^2 - b^2}} &\leq \varphi \leq \frac{\pi}{2} \end{aligned}$$

and compute the triple integral:

$$\begin{aligned}
\iiint_{\mathcal{R}} dV &= \int_{\tan^{-1}\frac{b}{\sqrt{a^2-b^2}}}^{\pi/2} \int_0^{2\pi} \left(\int_{b/\sin\varphi}^{a} \rho^2 \sin\varphi \, d\rho \right) d\theta \, d\varphi \\
&= \int_{\tan^{-1}\frac{b}{\sqrt{a^2-b^2}}}^{\pi/2} \int_0^{2\pi} \left(\frac{a^3}{3} - \frac{b^3}{\sin^3\varphi} \right) \sin\varphi \, d\theta \, d\varphi \\
&= 2\pi \int_{\tan^{-1}\frac{b}{\sqrt{a^2-b^2}}}^{\pi/2} \left(\frac{a^3}{3}\sin\varphi - \frac{b^3}{\sin^2\varphi} \right) d\varphi \\
&= 2\pi \left[-\frac{a^3}{3}\cos\varphi + b^3 \cot\varphi \right] \Bigg|_{\tan^{-1}\frac{b}{\sqrt{a^2-b^2}}}^{\pi/2} \\
&= 2\pi \left[\frac{a^3}{3} \frac{\sqrt{a^2-b^2}}{a} - b^3 \frac{\sqrt{a^2-b^2}}{b} \right] \\
&= \frac{2\pi}{3}(a^2-b^2)^{\frac{3}{2}}
\end{aligned}$$

where we used the triangle in Figure 2 to find the trig functions of $\varphi = \tan^{-1}(b/\sqrt{a^2-b^2})$. Recalling that we computed the volume of the solid above the xy plane, we finally get

$$V = \frac{4\pi}{3}(a^2-b^2)^{\frac{3}{2}}.$$

As a "reality check," notice that if $b - 0$ (no cylindrical hole) we get $V = \frac{4\pi}{3}a^3$, the usual formula for the volume of a sphere.