Math 213 - General Coordinates

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October 30, 2023

Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

Change of Variables

One variable: If

$$x = f(u), \quad dx = f'(u)\,du$$

then

$$\int_a^b g(x)\,dx = \int_c^d g(f(u))\,f'(u)\,du$$

where $f(c) = a$, $f(d) = b$

The factor $f'(u)$ measures how lengths are stretched under the map

$$u \mapsto f(u)$$

This transformation is a map from $\mathbb{R}$ to $\mathbb{R}$

Two Variables: (polar coordinates) If $\mathcal{R}$ is a region described by a polar rectangle $a \le r \le b$, $\alpha \le \theta \le \beta$, then

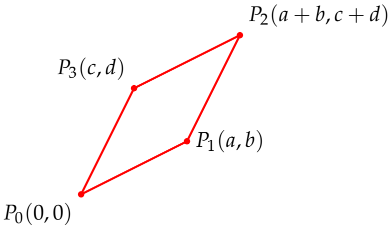
$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

The factor r measures how areas are stretched under the map

$$(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

This transformation is a map from $\mathbb{R}^2$ to $\mathbb{R}^2$

Interlude - Area of a Parallelogram



The area of the parallelogram is the magnitude of

$$\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix}$$

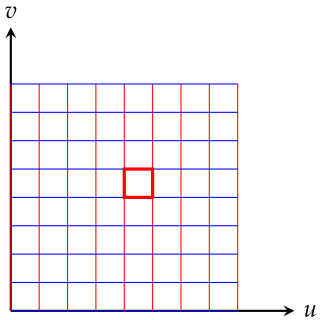
which is the absolute value of

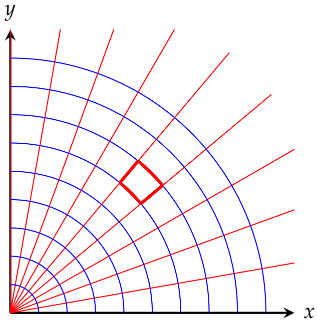
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Change of Variables

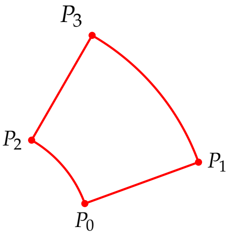
Let's consider a change of variables

$$x = x(u, v)$$
$$y = y(u, v)$$





Change of Variables



$$P_0 = (x(u,v), y(u,v))$$
$$P_1 = (x(u+\Delta u, v), y(u+\Delta u, v))$$
$$P_2 = (x(u, v+\Delta v), y(u, v+\Delta v))$$
$$P_3 = (x(u+\Delta u, v+\Delta v), y(u+\Delta u, v+\Delta v))$$

The small region in the xy plane is (approximately) a parallelogram with sides $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$:

$$\overrightarrow{P_0P_1} \simeq \left\langle \frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v) \right\rangle \Delta u$$

$$\overrightarrow{P_0P_2} \simeq \left\langle \frac{\partial x}{\partial v}(u,v), \frac{\partial y}{\partial v}(u,v) \right\rangle \Delta v$$

From the earlier determinant formula, this area is

$$\left| \det \begin{pmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial y}{\partial u}(u,v) \\ \frac{\partial x}{\partial v}(u,v) & \frac{\partial y}{\partial v}(u,v) \end{pmatrix} \right| \Delta u \, \Delta v$$

Area Element

The matrix

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u}(u,v) & \frac{\partial x}{\partial v}(u,v) \\ \frac{\partial y}{\partial u}(u,v) & \frac{\partial y}{\partial v}(u,v) \end{pmatrix}$$

is called the *Jacobian matrix* of the transformation.

For the change of variables

$$x = x(u,v), \quad y = y(u,v)$$

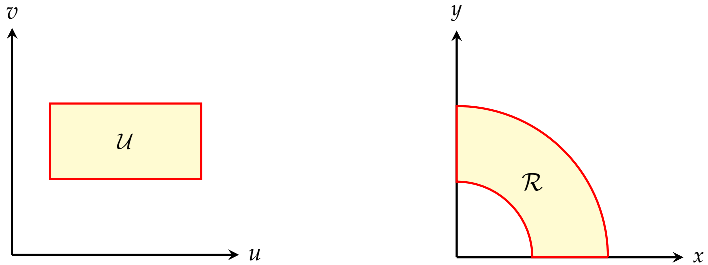
we have shown that

$$dA = \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

Integration with General Coordinates

Suppose that $\mathcal{U}$ is a region in the uv plane and that the transformation $x = x(u, v)$, $y = y(u, v)$ maps $\mathcal{U}$ to a region $\mathcal{R}$ in the xy plane. Then

$$\int_{\mathcal{R}} f(x, y)\, dA = \int_{\mathcal{U}} f(x(u, v), y(u, v)) \left| \det \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$



Coordinate Change Examples - Polar Coordinates

If $x(u, v) = u \cos v, y(u, v) = u \sin v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix}$$

$$= u$$

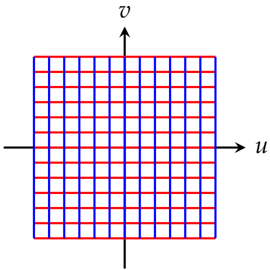
so

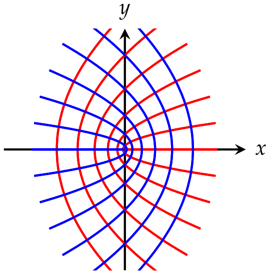
$$\boxed{dA = u\, du\, dv}$$

Puzzler #1 - Parabolic Coordinates

Parabolic coordinates are defined by

$$x(u,v) = \frac{u^2 - v^2}{2}, \quad y(u,v) = uv$$

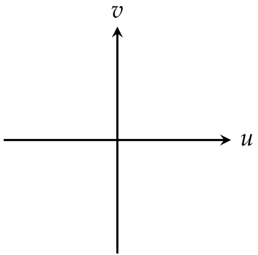


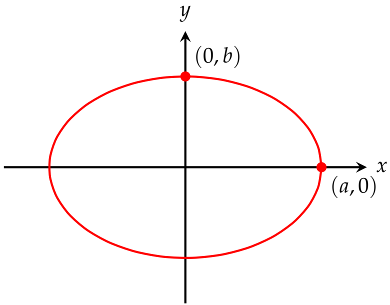


Find $\dfrac{\partial(x,y)}{\partial(u,v)}$ and find dA

Puzzler #2 - Scaled Coordinates

Suppose that $x = au$ and $y = bv$. What curve in the uv plane is mapped to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$?





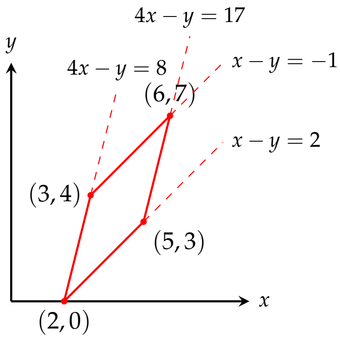
Using Coordinate Transformations

Using the transformation $x = au$, $y = bv$, find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Using Coordinate Transformations - Parallelogram

Find $\iint_{\mathcal{R}} (6x - 3y)\, dA$ if $\mathcal{R}$ is the parallelogram $(2,0)$, $(5,3)$, $(6,7)$, $(3,4)$. Use the transformation

$$x = \frac{1}{3}(v - u), \quad y = \frac{1}{3}(4v - u)$$



The figure at left shows why this change of variables works. If we pick

$$u = x - y$$
$$v = 4x - y$$

then the parallelogram is described by

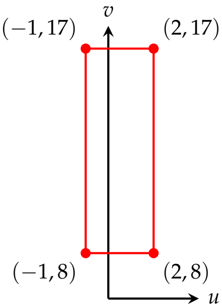
$$-1 \le u \le 2, \quad 8 \le v \le 17$$

From Paul's Online Math Notes, Problem 8 of Section 15.8

Using Coordinate Transformations - Parallelogram

Find $\iint_{\mathcal{R}} (6x - 3y)\, dA$ if $\mathcal{R}$ is the parallelogram $(2, 0)$, $(5, 3)$, $(6, 7)$, $(3, 4)$. Use the transformation

$$x = \frac{1}{3}(v - u), \quad y = \frac{1}{3}(4v - u)$$



If we solve

$$u = x - y$$
$$v = 4x - y$$

for x and y, we get the transformation above.

Now find the Jacobian of the coordinate transformation and compute the double integral!

From Paul's Online Math Notes, Problem 8 of Section 15.8

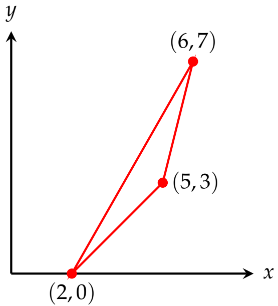
Puzzler #2 - Integrating over a Triangle

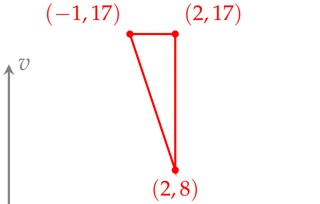
Evaluate $\iint_{\mathcal{R}} (6x - 3y)\, dA$ if R is the triangle with vertices $(2,0)$, $(5,3)$, $(6,7)$. Use the transformation

$$x = \frac{1}{3}(v-u), y = \frac{1}{3}(4v-u)$$

and recall that

$$u = x - y, \quad v = 4x - y.$$





From Paul's Online Math Notes, Problem 9 of Section 15.8

Three Dimensions

For transformations

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w)$$

the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \end{pmatrix}$$

and the volume element is

$$dV = \left| \det \left(\frac{\partial(x, y, z)}{\partial(u, v, w)} \right) \right| du\, dv\, dw$$

Three Dimensions

Suppose that $\mathcal{U}$ is a region in uvw space and that the transformation

$$x = x(u,v,w), \quad y = y(u,v,w), \quad z = z(u,v,w)$$

maps $\mathcal{U}$ to a region $\mathcal{R}$ in xyz space. Then

$$\int_{\mathcal{R}} f(x,y,z)\, dV = \int_{\mathcal{U}} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$

Reminders for the Week of October 30-November 3

- Homework C1 (triple Integrals) due October 30, 11:59 PM
- Homework C2 (cylindrical coordinates) due November 1, 11:59 PM
- Quiz 8 on triple integrals in Cartesian and cylindrical coordinates due November 2, 11:59 PM
- Homework C3 on triple integrals in spherical coordinates due at 11:59 PM