Math 213 - Dot Products

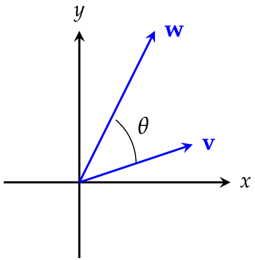
Peter Perry

August 25, 2023

Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

The Dot Product, Two Dimensions



If $\mathbf{v} = \langle a_1, a_2 \rangle$ and $\mathbf{w} = \langle b_1, b_2 \rangle$ are vectors, the *dot product* of $\mathbf{v}$ and $\mathbf{w}$ is given either by

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos\theta$$

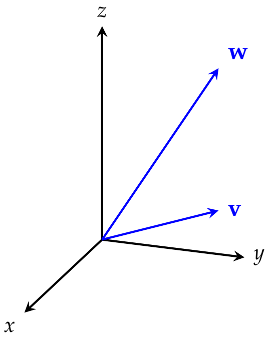
or by

$$\mathbf{v} \cdot \mathbf{w} = a_1 b_1 + a_2 b_2$$

Find $\mathbf{v} \cdot \mathbf{w}$ if:

- $\mathbf{v} = \langle 3, 5 \rangle$ and $\mathbf{w} = \langle -1, 2 \rangle$ Answer: $-3 + 10 = 7$
- $\mathbf{v} = \langle 1, 0 \rangle$ and $\mathbf{w} = \langle 3, 1 \rangle$ Answer: $3 + 0 = 3$
- $\mathbf{v} = \langle 1, 1 \rangle$ and $\mathbf{w} = \langle 2, -2 \rangle$ Answer: $2 + (-2) = 0$
- $|\mathbf{v}| = 3$, $|\mathbf{w}| = 5$ and $\theta = \pi/4$ Answer: $3 \cdot 5 \cdot \cos(\pi/4) = 15 \cdot (\sqrt{2}/2)$

The Dot Product, Three Dimensions



If

$$\mathbf{v} = \langle a_1, a_2, a_3 \rangle$$
$$\mathbf{w} = \langle b_1, b_2, b_3 \rangle$$

then the dot product of **v** and **w** is given either by

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}|\,\cos\theta$$

or

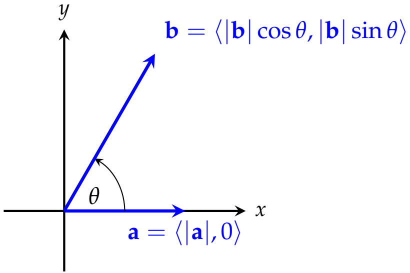
$$\mathbf{v} \cdot \mathbf{w} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Find $\mathbf{v} \cdot \mathbf{w}$ if:

- $\mathbf{v} = \langle 1, -2, 1 \rangle$ and $\mathbf{w} = \langle 1, 1, 1 \rangle$ Answer: $1 - 2 + 2 = 1$
- $|\mathbf{v}| = 2$, $|\mathbf{w}| = 5$, and $\theta = \pi/3$ Answer: $2 \cdot 5 \cdot \cos(\pi/3) = 5$

The Why of Dot Products

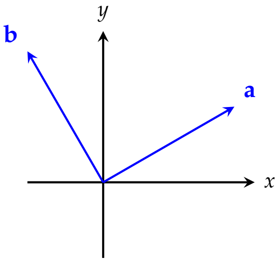
Why is $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$?



Here $\mathbf{a}$ points along the x-axis and $\mathbf{b}$ makes an angle θ with the x-axis

Use the "component" definition to find $\mathbf{a} \cdot \mathbf{b}$

Definition Break



Two vectors **a** and **b** are *perpendicular* if the angle between them is $\pi/2$

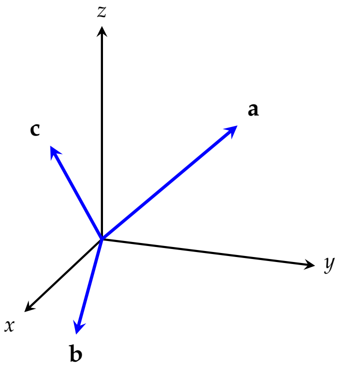
In other words, two vectors are perpendicular if they form a right angle.

If **a** and **b** are perpendicular we can also say that **a** and **b** are *orthogonal*.

Dot Product Properties

1. The dot product is a *scalar*
2. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
3. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
4. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
5. $s(\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
6. $\mathbf{0} \cdot \mathbf{a} = 0$
7. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$
8. If $\mathbf{a} \cdot \mathbf{b} = 0$ then either $\mathbf{a} = 0$, $\mathbf{b} = 0$ or $\mathbf{a}$ and $\mathbf{b}$ are perpendicular

Puzzler # 1 - Orthogonal Vectors



Which of the following vectors are orthogonal?

$$\mathbf{a} = \langle 3, 3, 3\rangle$$
$$\mathbf{b} = \langle 1, 0, -1\rangle$$
$$\mathbf{c} = \langle 2, 0, 2\rangle$$

Compute the dot products of pairs:

$\mathbf{a} \cdot \mathbf{b} = 3 - 3 = 0$ orthogonal

$\mathbf{b} \cdot \mathbf{c} = 2 - 2 = 0$ orthogonal

$\mathbf{c} \cdot \mathbf{a} = 6 + 6 = 0$ not orthogonal

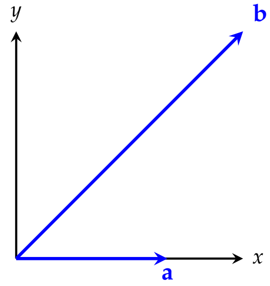
The Angle Between Two Vectors

Since

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$$

we can use the dot product to find the angle between two vectors:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$$



Find the angle between the vectors $\mathbf{a} = \langle 2, 0 \rangle$ and $\mathbf{b} = \langle 3, 3 \rangle$

Since

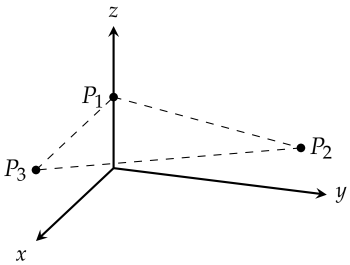
$$\mathbf{a} \cdot \mathbf{b} = 6, \quad |\mathbf{a}| = 2, \quad |\mathbf{b}| = 3\sqrt{2}$$

we get

$$\cos\theta = \frac{6}{2 \cdot 3\sqrt{2}} = \frac{1}{\sqrt{2}}$$

so $\theta = \pi/4$.

Puzzler #2 - Right Angles



Does the triangle with vertices

$$P_1 = (0,0,1)$$
$$P_2 = (1,3,1)$$
$$P_3 = (3,0,1)$$

have a right angle?

The sides of the triangle are given by vectors

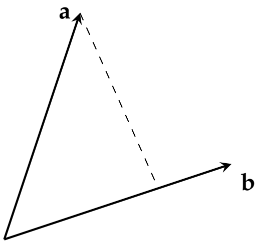
$$\overrightarrow{P_1P_2} = \langle 1,3,0\rangle, \quad \overrightarrow{P_2P_3} = \langle 2,-3,0\rangle, \quad \overrightarrow{P_3P_1} = \langle -3,0,0\rangle$$

Compute the dot products:

$$\overrightarrow{P_1P_2}\cdot\overrightarrow{P_2P_3} = 2-9 = -7, \quad \overrightarrow{P_2P_3}\cdot\overrightarrow{P_3P_1} = -6, \quad \overrightarrow{P_3P_1}\cdot\overrightarrow{P_1P_2} = -3$$

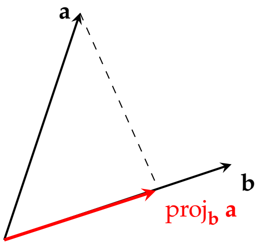
Since no dot product is zero, the triangle has no right angles.

Projections



Suppose that **a** and **b** are two vectors. Drop a perpendicular from the head of **a** to the line that contains **b**

Projections

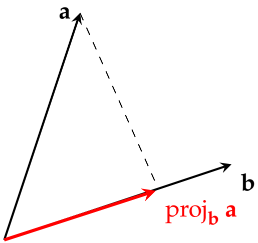


Suppose that **a** and **b** are two vectors. Drop a perpendicular from the head of **a** to the line that contains **b**

The *projection* of **a** onto **b** is the vector from the tail of **b** to the point where the perpendicular hits. It is denoted

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a}$$

Projections



Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two vectors. Drop a perpendicular from the head of $\mathbf{a}$ to the line that contains $\mathbf{b}$

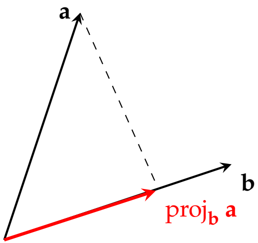
The *projection* of $\mathbf{a}$ onto $\mathbf{b}$ is the vector from the tail of $\mathbf{b}$ to the point where the perpendicular hits. It is denoted

$$\operatorname{proj}_{\mathbf{b}} \mathbf{a}$$

The *direction* of $\operatorname{proj}_{\mathbf{b}} \mathbf{a}$ is given by the unit vector $\mathbf{b}/|\mathbf{b}|$

The *signed length* of $\operatorname{proj}_{\mathbf{b}} \mathbf{a}$ is given by $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

Projections



Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two vectors. Drop a perpendicular from the head of $\mathbf{a}$ to the line that contains $\mathbf{b}$

The *projection* of $\mathbf{a}$ onto $\mathbf{b}$ is the vector from the tail of $\mathbf{b}$ to the point where the perpendicular hits. It is denoted

$$\text{proj}_{\mathbf{b}}\, \mathbf{a}$$

The *direction* of $\text{proj}_{\mathbf{b}}\, \mathbf{a}$ is given by the unit vector $\mathbf{b}/|\mathbf{b}|$

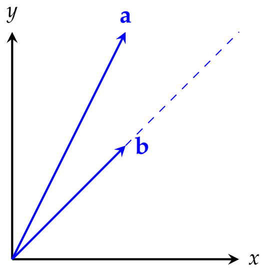
The *signed length* of $\text{proj}_{\mathbf{b}}\, \mathbf{a}$ is given by $\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

From these two facts we get the formula

$$\text{proj}_{\mathbf{b}}\, \mathbf{a} = \underbrace{\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}}_{\textit{length}} \; \underbrace{\frac{\mathbf{b}}{|\mathbf{b}|}}_{\textit{direction}}$$

Projections

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \underbrace{\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}}_{length}\ \underbrace{\frac{\mathbf{b}}{|\mathbf{b}|}}_{direction} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$



Find the projection of $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ onto $\mathbf{b} = \mathbf{i} + \mathbf{j}$

Since

$$\mathbf{a}\cdot\mathbf{b} = 3, \quad |\mathbf{b}| = \sqrt{2}$$

and

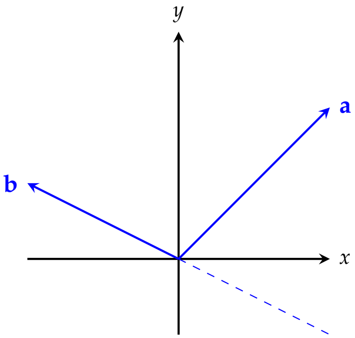
$$\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

we get

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right)$$

Projections

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = \underbrace{\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}}_{\textit{length}}\ \underbrace{\frac{\mathbf{b}}{|\mathbf{b}|}}_{\textit{direction}} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$



Find the projection of $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j}$ onto $\mathbf{b} = -2\mathbf{i} + \mathbf{j}$

Since

$$\mathbf{a}\cdot\mathbf{b} = -2, \quad |\mathbf{b}| = \sqrt{5}$$

and

$$\frac{\mathbf{b}}{|\mathbf{b}|} = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

we get

$$\text{proj}_{\mathbf{b}}\,\mathbf{a} = -\frac{2}{\sqrt{5}}\left(-\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}\right)$$

Determinant Rules

Next time, we'll introduce another multiplication of vectors, the *cross product* $\mathbf{v} \times \mathbf{w}$, which will be a *vector*.

We'll need some facts about 2×2 and 3×3 determinants.

The determinant of a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

This can be visualized as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \color{red}{a_{11}a_{22}} - \color{blue}{a_{12}a_{21}}.$$

Determinant Rules

We'll also need to compute 3×3 determinants.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

You can find some nice videos about 3×3 determinants from the Kahn Academy:

- Part 1: Standard method
- Part 2: Alternate Method (many students like this)

Determinant Practice

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Find the 2×2 determinant

$$\begin{vmatrix} 1 & -3 \\ 4 & 5 \end{vmatrix}$$

Find the 3×3 determinant

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & -1 & 2 \\ -3 & 2 & 0 \end{vmatrix}$$

The Cross Product - What it Is

The cross product of

$$\mathbf{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{w} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

is a new *vector* given by

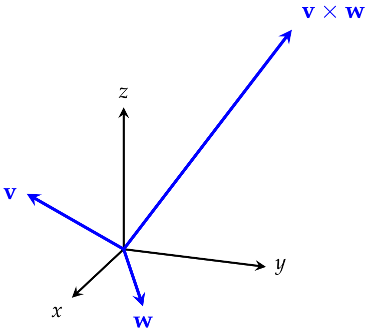
$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: Find the cross product of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j}$ by evaluating

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 3 & 0 \end{vmatrix}$$

The Cross Product – What it Does

The cross product of two vectors **v** and **w** is a new vector which is perpendicular to both **v** and **w** and points in a direction specified by the "right-hand rule"



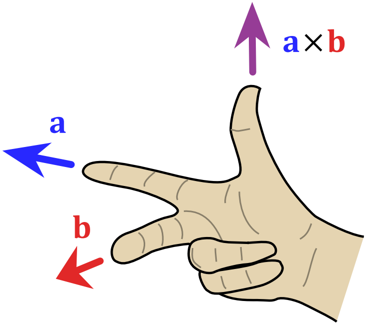
In this example,

$$\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$
$$\mathbf{w} = 2\mathbf{i} + 1\mathbf{j}$$

and

$$\mathbf{v} \times \mathbf{w} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

The Right-Hand Rule



The right-hand rule (see left) determines the direction of $\mathbf{a} \times \mathbf{b}$ from the directions of $\mathbf{a}$ and $\mathbf{b}$

Reminders for August 21-25 and August 28-September 1

- Friday 8/25 - Webwork A1 due at 11:59 PM
- Monday 8/28 - Read CLP 3.1.2 on Cross Products before class
- Wednesday 8/30 - Read CLP 3.1.4 on Equations of Planes before class
- Wednesday 8/30 - Webwork A2 due at 11:59 PM
- Quiz #1 due Thursday, August 31 at 11:59 PM
- Friday 9/1 - Read CLP 3.1.5 on Equations of Lines before class