Math 213 - Vector Fields

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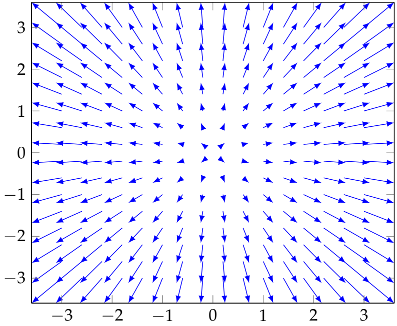
November 1, 2023

Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

A New Kind of Function



A *vector field* $\mathbf{v}(x, y)$ is a function

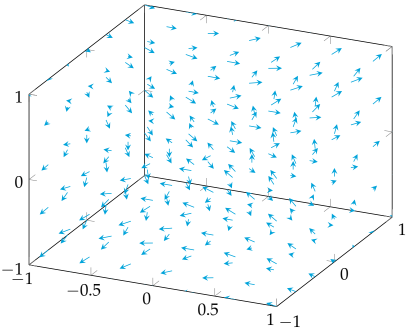
$$\mathbf{v}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

that associates to each point in the xy-plane a two-dimensional vector.

At left is a plot of the vector field

$$\mathbf{v}(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$$

A New Kind of Function



A *vector field* $\mathbf{v}(x, y, z)$ is a function

$P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$

that associates to each point in xyz space a three-dimensional vector

At left is a plot of the vector field

$$\mathbf{v}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

Why Vector Fields?

If $\mathbf{v}(x, y, z)$ is the velocity of a moving fluid at (x, y, z), then $\mathbf{v}(x, y, z)$ is called the *velocity field*

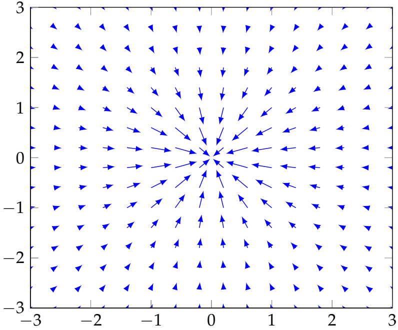
If $\mathbf{F}(x, y, z)$ is the force at position (x, y, z), then $\mathbf{F}(x, y, z)$ is called a *Force field*

For example,

$\mathbf{E}(x, y, z)$ is the *electric field* at a point (x, y, z) in space

$\mathbf{B}(x, y, z)$ is the *magnetic field* at a point (x, y, z) in space

Gravitational Fields



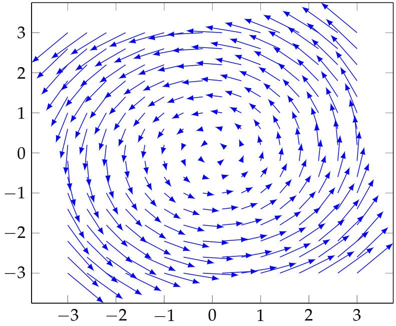
If M is the mass of the sun, then at a point (x, y, z), the sun's gravitational force on an object of mass m is

$$\mathbf{F}(x, y, z) = -\frac{GMm}{r^2}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right)$$

where G is *Newton's gravitational constant* and

$$r = \sqrt{x^2 + y^2 + z^2}.$$

Puzzler - The Vortex



At left is a plot of the vector field

$$\mathbf{v}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

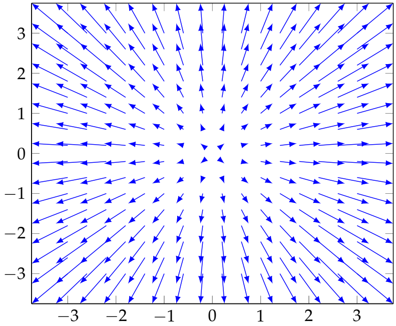
representing the velocity field of flow around a vortex.

Notice that

$$\mathbf{v}(x,y) \cdot \mathbf{i} = -y$$
$$\mathbf{v}(x,y) \cdot \mathbf{j} = x$$

Check that the direction of arrows is correct in each of the four quadrants!

Puzzler - The Twig



Imagine that $\mathbf{v}(x, y) = x\mathbf{i} + y\mathbf{j}$ is the velocity field of a fluid.

1. At time $t = 0$ you drop the twig at the point $(1, 1)$. Approximately where is the twig at $t = 0.01$?
2. At time $t = 0$ you drop the twig at the point $(0, 0)$. Where is the twig at $t = 0.01$?
3. At time $t = 0$ you drop the twig at the point $(0, 0)$. Where is the twig at $t = 10$?

From CLP 4-2.1, Problem 5

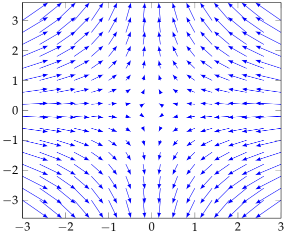
Puzzler - Mix and Match

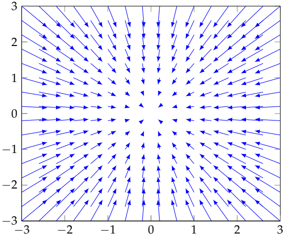
Can you match these vector fields with their field plots?

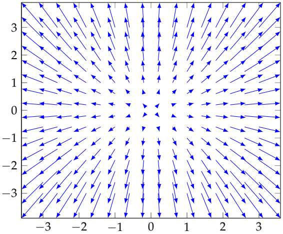
$\mathbf{v}(x,y) = -x\mathbf{i} - y\mathbf{j}$

$\mathbf{v}(x,y) = 2x\mathbf{i} + 3y\mathbf{j}$

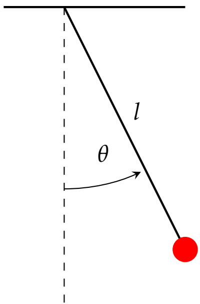
$\mathbf{v}(x,y) = -x\mathbf{i} + y\mathbf{j}$







Culture Break - Differential Equations and Vector Fields



The motion of a pendulum of length l is described by the equation

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

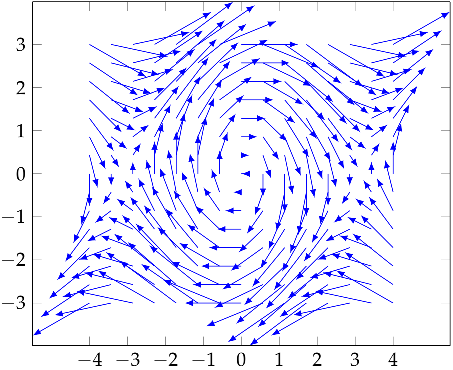
To study this equation, it is helpful to introduce the variables

$$x(t) = \theta(t)$$
$$y(t) = \theta'(t)$$

The equation of motion for the new variables is

$$x'(t) = y(t)$$
$$y'(t) = -\frac{g}{l}\sin(x(t))$$

Culture Break, Continued



The right-hand side of the equations

$$x'(t) = y(t)$$
$$y'(t) = -\frac{g}{l}\sin(x(t))$$

define a vector field (setting $g/l = 1$)

$$\mathbf{v}(x, y) = y\mathbf{i} - \sin(x)\mathbf{j}$$

Solution curves "follow the arrows."

Reminders for the Week of October 30-November 3

- Homework C2 (cylindrical coordinates) due November 1, 11:59 PM
- Quiz 8 on triple integrals in Cartesian and cylindrical coordinates due November 2, 11:59 PM
- Homework C3 on triple integrals in spherical coordinates due at 11:59 PM