Math 213 - Conservative Vector Fields

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Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

Conservative Vector Fields - Two Dimensions

A vector field $\mathbf{F}$ is conservative if there is a scalar function φ so that

$$\mathbf{F}(x,y) = (\nabla\varphi)(x,y).$$

The function φ is called a *potential* for the vector field $\mathbf{F}$.

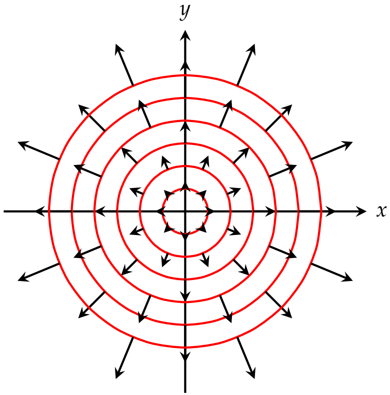
If $\mathbf{F}$ is a conservative vector field and C is a constant, the set of points (x,y) obeying $\varphi(x,y) = C$ is called an *equipotential curve*.

Example: If $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$:

$$\frac{\partial\varphi}{\partial x} = x, \quad \frac{\partial\varphi}{\partial y} = y$$

We can solve these equations to find $\varphi(x,y)$.

Vector Field and Potential



At left are equipotential curves and vectors for the vector field

$$\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$$

and

$$\varphi(x,y) = \frac{1}{2}\left(x^2 + y^2\right)$$

Conservative Vector Fields- Three Dimensions

A vector field **F** is *conservative* if there is a scalar function φ so that

$$\mathbf{F}(x, y, z) = (\nabla \varphi)(x, y, z)$$

The function φ is called a *potential* for the vector field **F**.

If **F** is a conservative vector field and C is a constant, the set of points (x, y, z) obeying $\varphi(x, y, z) = C$ is called an *equipotential surface*.

When is a Vector Field Conservative?

Two dimensions: If a vector field $\mathbf{F}(x,y)$ is conservative and

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

then

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y).$$

Proof: If $\mathbf{F}(x,y)$ is conservative,

$$P(x,y) = \frac{\partial \varphi}{\partial x}(x,y), \quad Q(x,y) = \frac{\partial \varphi}{\partial y}(x,y)$$

so

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial^2 \varphi}{\partial y \partial x}(x,y) = \frac{\partial^2 \varphi}{\partial x \partial y}(x,y) = \frac{\partial P}{\partial y}(x,y)$$

$\square$

Find the Potential if You Can

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y)$$

Determine if these vector fields may be conservative and, if so, find the potential.

- $\mathbf{F}(x,y) = (2x^3y^4 + x)\mathbf{i} + (2x^4y^3 + y)\mathbf{j}$

- $\mathbf{F}(x,y) = (2xe^{xy} + x^2ye^{xy})\mathbf{i} + (x^3e^{xy} + xy)\mathbf{j}$

When is a Vector Field Conservative?

Three dimensions: If a vector field

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

is conservative, then the *curl* of **F** is zero, where the curl is given by

$$(\nabla \times \mathbf{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

We'll come back to the proof later!

Aside: What on Earth is the Curl?

The curl of a vector field is one of three important differentiation operations for functions and vector fields. We'll meet it again when we study vector calculus.

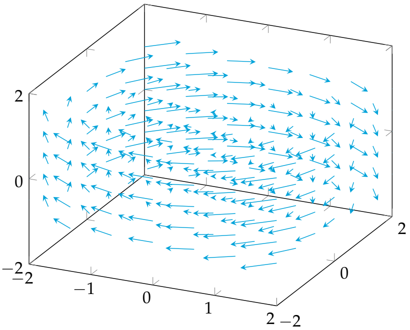
The curl measures the *vorticity* of a vector field. Its direction is the axis of rotation and its magnitude is the speed of rotation.

A good way to remember the formula for the curl is:

$$(\nabla \times \mathbf{F})(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}$$

Example: Find the curl of the vector field $\mathbf{F}(x,y,z) = y\mathbf{i} - x\mathbf{j}$.

What the Curl Measures



If $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j}$ then

$$(\nabla \times \mathbf{F})(x, y, z) = -2\mathbf{k}$$

Find the Potential, if You Can

$$(\nabla \times F) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Determine whether

$$\mathbf{F}(x, y, z) = (2xy)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$$

is conservative and, if so, find a potential for the vector field.

Reminders for the Week of October 30-November 3

- Homework C3 on triple integrals in spherical coordinates due at 11:59 PM