# Math 213 - Line Integrals

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# Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

# Work

In Physics, the work done by a constant force $\mathbf{F}$ over a distance $\mathbf{d}$ is given by

$$W = \mathbf{F} \cdot \mathbf{d}$$

What if the force varies with position and works over a path $\mathbf{r}(t)$ for $a \leq t \leq b$?

Over an interval $\Delta t$, the force $\mathbf{F}(\mathbf{r}(t))$ does work

$$\begin{aligned} \Delta W &= \mathbf{F}(\mathbf{r}(t)) \cdot (\Delta \mathbf{r}) \\ &= \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \Delta t \end{aligned}$$

So the work done between $t = a$ and $t = b$ is

$$W = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

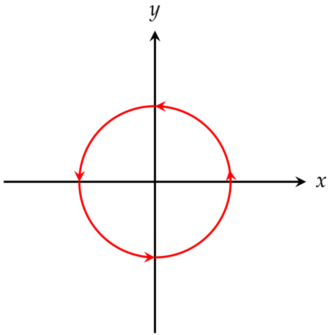
This integral is an example of the *line integral of a vector field*.

The line integral of a vector field $\mathbf{F}$ around a curve $\mathcal{C}$ is denoted

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

# Puzzler # 1

$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$




Let

$$\mathbf{F}(x, y) = 2y\mathbf{i} + 3x\mathbf{j}.$$

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$ if $C$ is the circle of radius 1 with center $(0, 0)$ oriented counterclockwise.

# Line Integrals

If $\mathcal{C}$ is the parametrized path $\mathbf{r}(t)$ with $a \le t \le b$ and

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

then

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{\mathcal{C}} P\,dx + Q\,dy + R\,dz \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}(t)\,dt\end{aligned}$$

For vector fields

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

and paths in the $xy$ plane,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{\mathcal{C}} P\,dx + Q\,dy \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}(t)\,dt.\end{aligned}$$

# Puzzler # 2

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}(t)\,dt$$

Find

$$\int_C \left( -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \right)$$

if $C$ is the circle of radius 1 with center $(0,0)$, oriented counterclockwise. Remember to:

- Parameterize the path
- Identify the vector field
- Use the formula above

# Conservative Vector Fields

Recall that a vector field $\mathbf{F}(x, y)$ is called *conservative* if there is a function $\varphi(x, y)$ so that

$$\mathbf{F}(x,y) = (\nabla\varphi)(x,y) = \frac{\partial\varphi}{\partial x}(x,y)\mathbf{i} + \frac{\partial\varphi}{\partial y}(x,y)\mathbf{j}.$$

What is the line integral of a conservative vector field?

Suppose that $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \leq t \leq b$ is a path from $\mathbf{r}(a) = P_0$ to $\mathbf{r}(b) = P_1$ Then

$$\mathbf{F}(\mathbf{r}(t)) = \frac{\partial\varphi}{\partial x}(x(t),y(t))\mathbf{i} + \frac{\partial\varphi}{\partial y}(x(t),y(t))\mathbf{j}.$$

So

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} &= \frac{\partial\varphi}{\partial x}(x(t),y(t))\, x'(t) + \frac{\partial\varphi}{\partial y}(x(t),y(t))\, y'(t) \\ &= \frac{d}{dt}\left(\varphi(x(t),y(t),z(t))\right)\end{aligned}$$

# Conservative Vector Fields

If $\mathbf{F}(x, y)$ is conservative and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a path from $P_0$ to $P_1$:

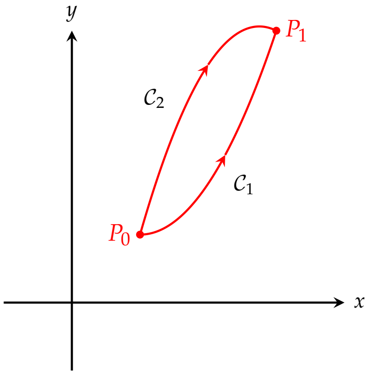
$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} &= \frac{\partial \varphi}{\partial x}(x(t), y(t))\, x'(t) + \frac{\partial \varphi}{\partial y}(x(t), y(t))\, y'(t) \\ &= \frac{d}{dt}\left(\varphi(x(t), y(t))\right) \end{aligned}$$

Then

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt \\ &= \int_a^b \frac{d}{dt}\left(\varphi(x(t), y(t))\right)\, dt \\ &= \varphi(P_1) - \varphi(P_0) \end{aligned}$$

The result is the same for *any* path from $P_0$ to $P_1$

# Conservative Vector Fields




If $\mathbf{F} = \nabla\varphi$ is a conservative vector field, then

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \varphi(P_1) - \varphi(P_0)$$

We also get the same result for any path beginning at $P_0$ and ending at $P_1$.

# Interlude

If $C_1$ is a path from $P_0$ to $P_1$ and $C_2$ is a path from $P_1$ to $P_0$ obtained by reversing $\mathcal{C}_1$, then

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$$

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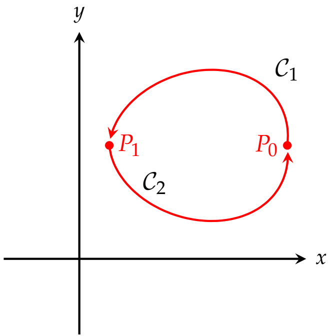
$$\mathcal{C}_1 : \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \leq t \leq b$$

$$\begin{aligned} \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} &= \int_b^a \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt \\ &= -\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt \\ &= -\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} \end{aligned}$$

# Integrating along a Closed Curve

If a path $\mathcal{C}$ is parametrized by $\mathbf{r}(t), a \leq t \leq b$ and $\mathbf{r}(a) = \mathbf{r}(b)$, then the path is a *closed path*, and $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is sometimes denoted $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.




Suppose that $\mathcal{C}$ is a closed path and that $\mathbf{F}$ is a *conservative* vector field. Find

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathcal{C}$ is the closed path consisting of $\mathcal{C}_1$ from $P_0$ to $P_1$ and $\mathcal{C}_2$ from $P_1$ back to $P_0$

# Path Independence

A line integral is *path independent* between points $P_0$ and $P_1$ if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is the same for *any* path connecting beginning at $P_0$ and ending at $P_1$.

Conservative vector fields are path independent since, for any path $\mathcal{C}$ between $P_0$ and $P_1$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(P_1) - \varphi(P_0).$$

**Theorem** If the vector field $\mathbf{F}$ is defined and continuous over all of $\mathbb{R}^2$ (or $\mathbb{R}^3$) and $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path between $P_0$ and $P_1$, then for any other point $P_0'$, and $P_1'$, $\int_{C'} \mathbf{F} \cdot d\mathbf{r}$ is path independent for any path $\mathcal{C}'$ connecting $P_0'$ and $P_1'$.

# Path Independence

**Theorem** Suppose that $\mathbf{F}$ is a vector field defined and continuous over all of $\mathbb{R}^2$ (or $\mathbb{R}^3$). The following are equivalent:

(a) $\mathbf{F}$ is conservative. That is, there is a function $\varphi$ so that $\mathbf{F} = \nabla \varphi$.

(b) $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path $\mathcal{C}$.

(c) The integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is path-independent. That is, for any points $P_0$ and $P_1$, $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{r}$ for any paths $\mathcal{C}$ and $\mathcal{C}'$ from $P_0$ to $P_1$.

Proof:

(a) $\Rightarrow$ (b): we showed this two slides ago!

(b) $\Rightarrow$ (c): we'll show this in class

(c) $\Rightarrow$ (a): we'll show how to get the function $\varphi$ in class

# Puzzler #3

Find a potential for the vector field

$$\mathbf{F}(x,y,z) = (z + e^y)\mathbf{i} + (xe^y - e^z \sin y)\mathbf{j} + (x + e^z \cos y)\mathbf{k}$$

and evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ for any path $\mathcal{C}$ between $P_0 = (0,0,0)$ and $P_1 = (1,\pi,0)$.

# Reminders for the Week of November 6-10

- Homework C4 is due tonight at 11:59 PM
- Homework C5 is due Wednesday night at 11:59 PM
- Quiz #9 is due Thursday at 11:59 PM
- Homework C6 is due Friday night at 11:59 PM
- Exam 3, covering Webworks B7-B8 and C1-C6, takes place on Wednesday, November 15
- Apply *no later than 5 PM on Friday, November 10* if you need to take an alternate exam and are not working with the DRC