Math 213 - Parametrized Surfaces

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November 8, 2023

Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

Important Notice

The material we cover today, November 10, and November 13 will *not* be covered on Exam 3. Exam 3 covers Double and Triple integrals, vector fields, conservative vector fields, and line integrals.

Now is a good time to begin reviewing for Exam 3!

Parametrized Surfaces

So far we've seen surfaces in three dimensions as:

- Graphs of functions $z = f(x, y)$
- Solution sets of equations like $x^2 + y^2 + z^2 = 1$ or $x^2 - y^2 + z^2 = 1$
- Level sets of functions of three variables

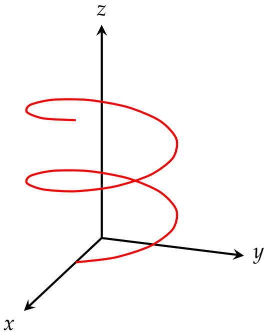
We're going to introduce a new way of describing surfaces through *parametrization*

Preview

A *curve* is described by a vector function with one parameter, t, like:

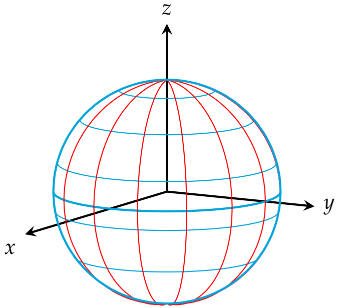
$$\mathbf{r}(t) = \langle \cos(t), \sin(t), t \rangle$$

for $0 \le t \le 4\pi$

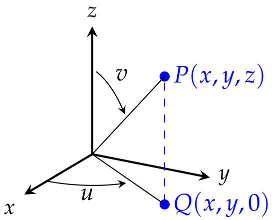


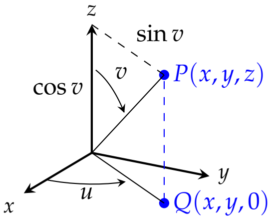
A *surface* can be described by a vector function with two parameters, like

$$\mathbf{r}(u, v) = \langle \sin(u)\cos(v), \sin(u)\sin(v), \cos(u) \rangle$$



Parametrizing the Sphere





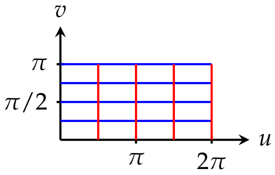
If P is a point on the sphere $x^2 + y^2 + z^2 = 1$:

- Let Q be the projection of P onto the xy plane
- Let v be the angle between $\overrightarrow{OP}$ and the z axis
- Let u be the angle between $\overrightarrow{OQ}$ and the x axis

In terms of u and v:

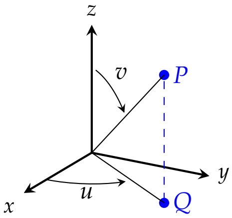
$$x = \sin v \cos u$$
$$y = \sin v \sin u$$
$$z = \cos v$$

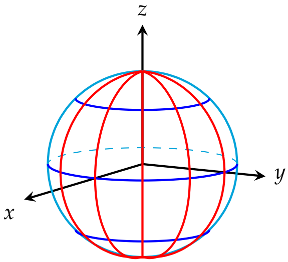
Parametrizing the Sphere



$$x = \sin v \cos u$$
$$y = \sin v \sin u$$
$$z = \cos v$$

$$0 \leq u \leq 2\pi$$
$$0 \leq v \leq \pi$$

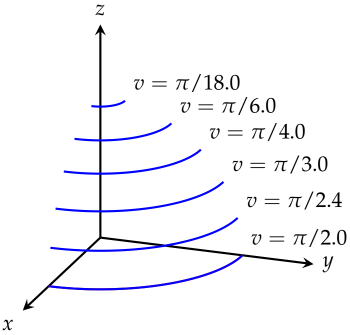




Parametrizing the Sphere

Let's look at this parametrization in detail:

$$x = \sin v \cos u$$
$$y = \sin v \sin u$$
$$z = \cos v$$



For $0 \leq u, v \leq \pi/2$ we get the part of the sphere in the first octant

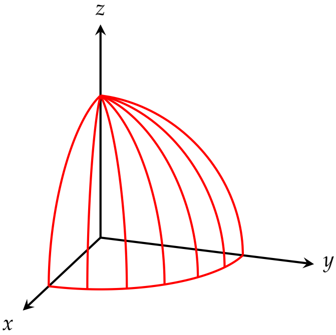
Lines of constant v are parallels of latitude:

$v = \pi/2,\ 5\pi/12,\ \pi/3,\ \pi/4,\ \pi/6,\ \pi/18$

Parametrizing the Sphere

Let's look at this parametrization in detail:

$$x = \sin v \cos u$$
$$y = \sin v \sin u$$
$$z = \cos v$$

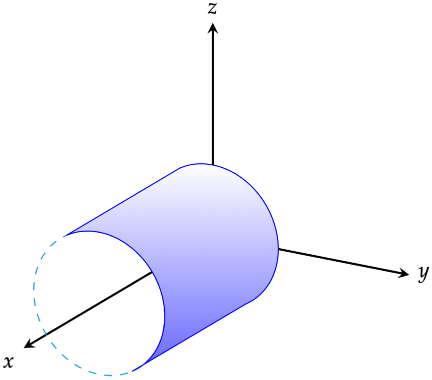


For $0 \leq u, v \leq \pi/2$ we get the part of the sphere in the first octant

Lines of constant u are parallels of longitude:

$u = 0,\ \pi/12,\ \pi/6,\ \pi/4,\ \pi/3,\ 5\pi/12,\ \pi/2$

Puzzler # 1



Find a parametrization for the cylinder

$$y^2 + z^2 = 1$$

- Choose two parameters
- Find a vector function of the two parameters that traces out the cylinder

$$\mathbf{r}(x, \theta) = \langle x, \cos\theta, \sin\theta \rangle$$

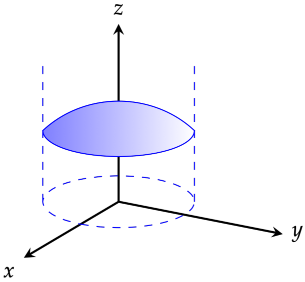
where

$$-\infty < x < \infty$$

and

$$0 \le \theta < 2\pi$$

Puzzler #2



Find a parametrization for the part of the sphere

$$x^2 + y^2 + z^2 = 2$$

that lies inside the cylinder

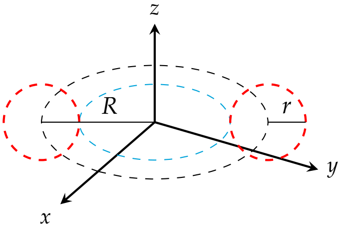
$$x^2 + y^2 = 1$$

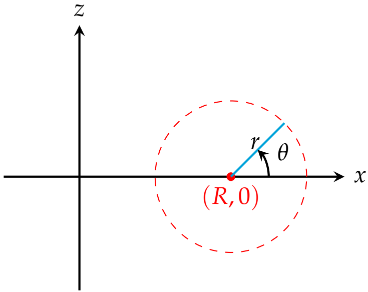
and above the xy plane.

- Choose two parameters
- Find a vector function of the two parameters that traces out the surface

$\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, \sqrt{2 - r^2}\rangle$ where $0 \leq r < 1$ and $0 \leq \theta \leq 2\pi$

Parametrizing the Torus





A torus is a circle of radius r revolved the origin in a circle of radius R

Strategy:

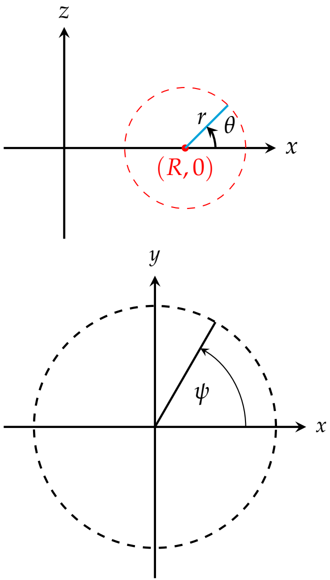
- Parametrize the intersection of the surface with the xz plane
- Obtain the torus by rotation

In the xz-plane:

$$x = R + r\cos\theta$$
$$z = r\sin\theta$$

where $0 \le \theta \le 2\pi$

Parametrizing the Torus



In the xz plane we found

$$
\begin{aligned}
x &= R + r\cos\theta \\
y &= 0 \\
z &= r\sin\theta
\end{aligned}
$$

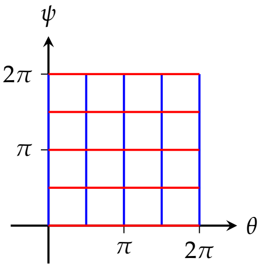
Now rotate by an angle ψ in the xy plane:

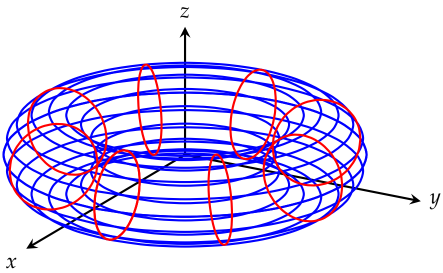
$$
\begin{aligned}
x &= (R + r\cos\theta)\cos\psi \\
y &= (R + r\cos\theta)\sin\psi \\
z &= r\sin\theta
\end{aligned}
$$

where $0 \le \psi \le 2\pi$.

Parametrizing the Torus

$$x = (R + r\cos\theta)\cos\psi$$
$$y = (R + r\cos\theta)\sin\psi$$
$$z = r\sin\theta$$





Tangent Planes: Preview

Next time we'll see how to find the *tangent plane* to a parameterized surface. Given a parametrization

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k},$$

we can find two tangent vectors to the surface at $\mathbf{r}(a,b)$ by finding

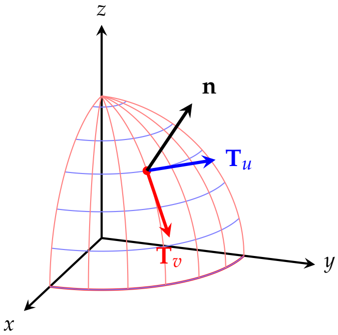
$$\mathbf{T}_u(a,b) = \frac{\partial x}{\partial u}(a,b)\mathbf{i} + \frac{\partial y}{\partial u}(a,b)\mathbf{j} + \frac{\partial z}{\partial u}(a,b)\mathbf{k}$$
$$\mathbf{T}_v(a,b) = \frac{\partial x}{\partial v}(a,b)\mathbf{i} + \frac{\partial y}{\partial v}(a,b)\mathbf{j} + \frac{\partial z}{\partial v}(a,b)\mathbf{k}$$

From these tangent vectors we can get a normal vector to the tangent plane

$$\mathbf{n}(a,b) = \mathbf{T}_u(a,b) \times \mathbf{T}_v(a,b)$$

Tangent Planes: Preview

Here's how it works for the sphere, visually:



The vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are tangent to the surface, and their cross product is normal to the surface

Reminders for the Week of November 6-10

- Homework C5 is due tonight at 11:59 PM
- Quiz #9 is due Thursday at 11:59 PM
- Homework C6 is due Friday night at 11:59 PM
- Exam 3, covering Webworks B7-B8 and C1-C6, takes place on Wednesday, November 15
- Apply *no later than 5 PM on Friday, November 10* if you need to take an alternate exam and are not working with the DRC