Math 213 - Tangent Planes

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November 10, 2023

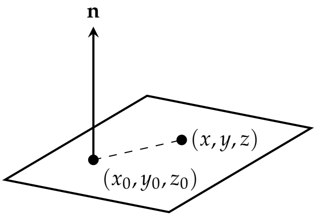
Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

Important Notice

The material we cover on November 8, today, and November 13 will *not* be covered on Exam 3. Exam 3 covers Double and Triple integrals, vector fields, conservative vector fields, and line integrals.

Planes



The equation of a plane is determined by

- A point (x_0, y_0, z_0) on the plane
- A vector $\mathbf{n}$ normal to the plane

$$\mathbf{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Tangent Planes

We've already seen:

- Tangent planes to the graph of a function $f(x, y)$
- Tangent planes to the level surface of a function $g(x, y, z)$

Tangent to the graph of a function $f(x, y)$

Point on the plane:

$$(a, b, f(a, b))$$

Normal:

$$\mathbf{n} = -\frac{\partial f}{\partial x}(a, b)\mathbf{i} - \frac{\partial f}{\partial y}(a, b)\mathbf{j} + \mathbf{k}$$

Tangent to the level surface $g(x, y, z) = C$

Point on the plane: (a, b, c) with $g(a, b, c) = C$

Normal:

$$\mathbf{n} = (\nabla g)(a, b, c)$$

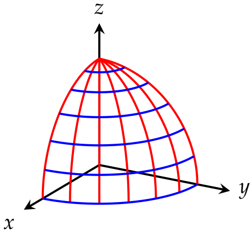
Parametrized Surfaces

A parametrized surface is described by a vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

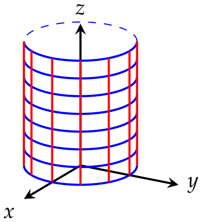
The sphere of radius 3:

$$x = 3\sin v \cos u$$
$$y = 3\sin v \sin u$$
$$z = 3\cos v$$



The cylinder of radius 1 with the z axis for an axis of symmetry:

$$x(u,v) = \cos u$$
$$y(u,v) = \sin u$$
$$z(u,v) = v$$



Tangent Plane to a Parametrized Surface

If a parametrized surface is given by a vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

two vectors tangent to the surface at $\mathbf{r}(a,b)$ are

$$\mathbf{T}_u(a,b) = \frac{\partial x}{\partial u}(a,b)\mathbf{i} + \frac{\partial y}{\partial u}(a,b)\mathbf{j} + \frac{\partial z}{\partial u}(a,b)\mathbf{k}$$
$$\mathbf{T}_v(a,b) = \frac{\partial x}{\partial v}(a,b)\mathbf{i} + \frac{\partial y}{\partial v}(a,b)\mathbf{j} + \frac{\partial z}{\partial v}(a,b)\mathbf{k}$$

and a normal to the tangent plane is given by

$$\mathbf{n} = \mathbf{T}_u(a,b) \times \mathbf{T}_v(a,b)$$

Puzzler #1

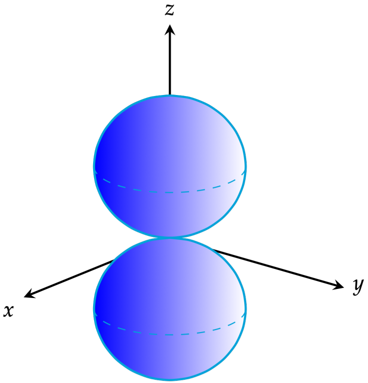
Are the surfaces

$$x^2 + y^2 + (z-1)^2 = 1$$

and

$$x^2 + y^2 + (z+1)^2 = 1$$

tangent to each other at $(0, 0, 0)$?



Puzzler # 2

$$\mathbf{T}_u(a,b) = \frac{\partial x}{\partial u}(a,b)\mathbf{i} + \frac{\partial y}{\partial u}(a,b)\mathbf{j} + \frac{\partial z}{\partial u}(a,b)\mathbf{k}$$

$$\mathbf{T}_v(a,b) = \frac{\partial x}{\partial v}(a,b)\mathbf{i} + \frac{\partial y}{\partial v}(a,b)\mathbf{j} + \frac{\partial z}{\partial v}(a,b)\mathbf{k}$$

$$\mathbf{n} = \mathbf{T}_u(a,b) \times \mathbf{T}_v(a,b)$$

Find the equation of the tangent plane to the cylinder

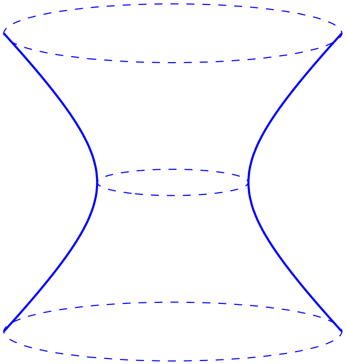
$$\mathbf{r}(u,v) = \cos u\mathbf{i} + \sin u\mathbf{j} + v\mathbf{k}$$

at the point

$$\mathbf{r}(\pi/4,2) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + 2\mathbf{k}$$

Puzzler #3

Find the tangent planes to the surface $x^2 + y^2 - z^2 = 1$ for any point (x_0, y_0, z_0) on the surface



This surface is a level surface of the function

$$g(x, y, z) = x^2 + y^2 - z^2$$

Puzzler #4

Find all points on the hyperboloid $z^2 = 4x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $2x + y - z = 0$

Puzzler #5

Find the equation of the tangent plane to the surface

$$\mathbf{r}(u,v) = \langle -4u, 2u^2 + 3v, v^2 \rangle$$

at the point $(4, -1, 1)$ (where $u = -1, v = -1$)

Preview: Surface Integrals

When a curve $\mathcal{C}$ is parametrized by a function $\mathbf{r}(t)$ over an interval $[a,b]$, we can compute the *line integral* of a scalar function or a vector field over the curve:

$$\iint_{\mathcal{C}} f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt$$

$$\iint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_a^b F(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt$$

When a surface $\mathcal{S}$ is parametrized by a function $\mathbf{r}(u,v)$ over a domain $\mathcal{D}$, we can compute the *surface integral* of a scalar function or a vector field over that surface:

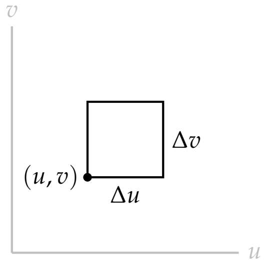
$$\iint_{\mathcal{S}} f(x,y,z)\,dA = \iint_{\mathcal{D}} f(x(u,v),y(u,v),z(u,v))\,dA$$

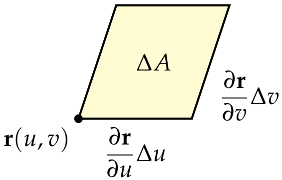
$$\iint_{S} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\mathcal{D}} \mathbf{F}(\mathbf{r}(u,v))\cdot\mathbf{n}\,dA$$

where $\mathbf{n}$ is the unit normal to the surface, and $dA = \left|\frac{\partial \mathbf{r}}{\partial u}\times\frac{\partial \mathbf{r}}{\partial v}\right|\,du\,dv$

Preview: Surface Area

If a surface is parameterized by a vector function $\mathbf{r}(u,v)$, we can find the area of a small piece of surface near $\mathbf{r}(u,v)$ obtained by making small changes in u and v:





$$\Delta A = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| \Delta u \, \Delta v$$

or

$$dA = \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du \, dv$$

Preview: Surface Area

If $\mathcal{S}$ is a surface parametrized by $\mathbf{r}(u, v)$ with domain $\mathcal{D}$, then the total area of the surface is

$$\iint_{\mathcal{S}} dA = \iint_{D} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv$$

Preview: Surface Area

$$\iint_S dA = \iint_D \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du\, dv$$

A hemisphere above the xy plane of radius a and centered at $(0,0,0)$ is parametrized by

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{a^2 - r^2} \rangle$$

for $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$. Find the surface area of the hemisphere.

Reminders for the Week of November 6-10

- Homework C6 is due tonight at 11:59 PM
- Exam 3, covering Webworks B7-B8 and C1-C6, takes place on Wednesday, November 15
- Apply *no later than 5 PM today* if you need to take an alternate exam and are not working with the DRC