Math 213 - Surface Integrals

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November 13, 2023

Unit C: Multiple Integrals

- October 13 - Double Integrals
- October 16 - Double Integrals in Polar Coordinates
- October 20 - Triple Integrals
- October 25 - Triple Integrals, Cylindrical Coordinates
- October 27 - Triple Integrals, Spherical Coordinates
- October 30 - Triple Integrals, General Coordinates
- November 1 - Vector Fields
- November 3 - Conservative Vector Fields
- November 6 - Line integrals
- November 8 - Parametrized Surfaces
- November 10 - Tangent Planes to Surfaces
- November 13 - Surface Integrals
- November 15 - Exam III Review

Important Notice

The material we cover on November 8, today, and November 13 will *not* be covered on Exam 3. Exam 3 covers Double and Triple integrals, vector fields, conservative vector fields, and line integrals.

Preview

Today we'll define two new kinds of integrals for functions and vector fields over a surface S in three-dimensional space.

(1) The surface integral of a function $f(x, y, z)$ over a surface S:

$$\int_S f \, dA$$

where dA is the differential surface area

(2) The surface integral of a vector field $\mathbf{F}(x, y, z)$ over a surface S:

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dA$$

where $\mathbf{n}$ is the outward unit normal to the surface

We'll consider:

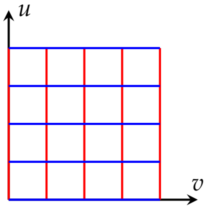
1. Parametrized surfaces
2. Surfaces defined as graphs

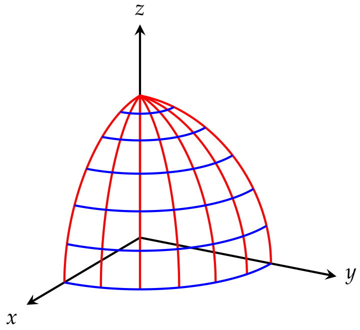
Parametrized Surfaces

Recall that a parametrized surface is described by a vector function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

that maps a domain U in the uv plane onto a surface S in three-dimensional space

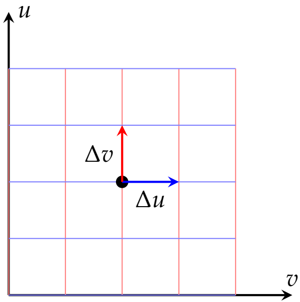


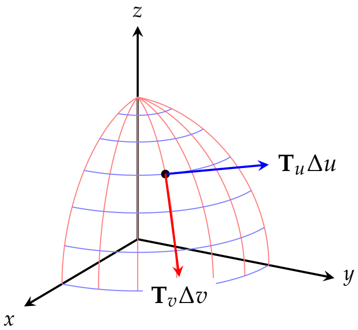


Parametrized Surfaces

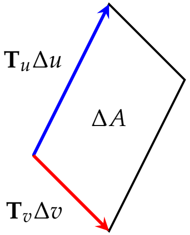
At each point $\mathbf{r}(u,v)$ on the surface, there are two tangent vectors

$$\mathbf{T}_u = \frac{\partial \mathbf{r}}{\partial u}(u,v), \quad \mathbf{T}_v = \frac{\partial \mathbf{r}}{\partial v}(u,v)$$





Area Element



The area of the parallelogram is

$$\Delta A = |\mathbf{T}_u \times \mathbf{T}_v| \; \Delta u \Delta v$$

so the area element is

$$\boxed{dA = |\mathbf{T}_u \times \mathbf{T}_v| \; du\, dv}$$

A unit normal is given by

$$\mathbf{n} = \frac{\mathbf{T}_u \times \mathbf{T}_v}{|\mathbf{T}_u \times \mathbf{T}_v|}$$

so

$$\boxed{\mathbf{n}\, dA = (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv}$$

Surfaces Integrals over Parametrized Surfaces

If $F(x, y, z)$ is a function defined in a neighborhood of a parametrized surface S with

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in U$$

then

$$\iint_S F(x, y, z)\, dA = \iint_U F(\mathbf{r}(u, v))\, |\mathbf{T}_u \times \mathbf{T}_v|\, du\, dv$$

If $\mathbf{F}(x, y, z)$ is a vector field defined in the neighborhood of S, then

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_U \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

Puzzler #1

$$\iint_S F(x,y,z)\,dA = \iint_U F(\mathbf{r}(u,v))\,|\mathbf{T}_u \times \mathbf{T}_v|\,du\,dv$$

Find the area of the hemisphere of radius a parametrized by

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + \sqrt{a^2 - r^2}\,\mathbf{k}$$

for $0 \le r \le a$ and $0 \le \theta \le 2\pi$.

Some helps:

$$\mathbf{T}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + \frac{r}{\sqrt{a^2 - r^2}}\,\mathbf{k}$$

$$\mathbf{T}_\theta = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}$$

Interlude: Integrating over a Sphere

A sphere of radius a is parametrized by

$$\mathbf{r}(u, v) = a \sin v \cos u \, \mathbf{i} + a \sin v \sin u \, \mathbf{j} + a \cos v \, \mathbf{j}.$$

For the sphere

$$\begin{aligned} \mathbf{T}_u &= \langle -a \sin v \sin u, a \sin v \cos u, 0 \rangle \\ \mathbf{T}_v &= \langle a \cos v \cos u, a \cos v \cos u, -a \sin v \rangle \end{aligned}$$

The normal vector is

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin v \sin u & a \sin v \cos u & 0 \\ a \cos v \cos u & a \cos v \sin u & -a \sin v \end{vmatrix}$$

or

$$\mathbf{n} = -(a^2 \sin v) \langle \sin v \cos u, \sin v \sin u, \cos v \rangle$$

The *outward* normal is

$$\mathbf{n} = (a^2 \sin v) \langle \sin v \cos u, \sin v \sin u, \cos v \rangle$$

Interlude: Integrating over a Sphere

For a sphere of radius a:

$$dA = a^2 \sin v \, du \, dv$$

and

$$\mathbf{n} \, dA = (a^2 \sin v) \langle \sin v \cos u, \sin v \sin u, \cos v \rangle \, du \, dv$$

or equivalently

$$\mathbf{n} dA = (a \sin v) \langle x, y, z \rangle \, du \, dv$$

if (x, y, z) is a point on the sphere.

Puzzler #2

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA = \iint_U \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

$$\mathbf{n}\, dA = (a \sin v)\langle x, y, z\rangle\, du\, dv$$

If S is a surface and $\mathbf{F}$ is a vector field defined in a neighborhood of S, then the *flux* of the vector field through S is

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS$$

where $\mathbf{n}$ is the *outward* unit normal. Find the flux of the vector field

$$\mathbf{F}(x,y,z) = (x^2 + y^2 + z^2)^n (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

out of the sphere

$$x^2 + y^2 + z^2 = a^2$$

Integrating Over a Graph

Suppose $f(x,y)$ is defined over a domain D in the xy plane and we want to integrate over the surface $z = f(x,y)$. We can parametrize the surface as

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + f(x,y)\mathbf{k}$$

so

$$\mathbf{T}_x = \left\langle 1, 0, \frac{\partial f}{\partial x} \right\rangle$$

$$\mathbf{T}_y = \left\langle 0, 1, \frac{\partial f}{\partial y} \right\rangle$$

$$\mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left(-\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(-\frac{\partial f}{\partial y}\right)\mathbf{j} + \mathbf{k}$$

Integrating Over a Graph

$$\mathbf{T}_x \times \mathbf{T}_y = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

If a surface S is given by $z = f(x,y)$ for (x,y) in a domain D, then

$$\boxed{dA = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx\, dy}$$

and

$$\boxed{\mathbf{n}\, dA = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle dx\, dy}$$

Integrating over a Graph

If a surface S is given by $z = f(x,y)$ for (x,y) in a domain D, then

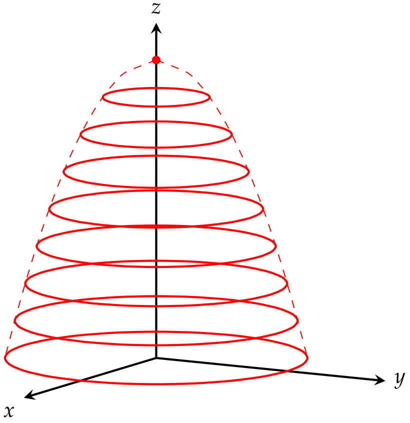
$$\iint_S G(x,y,z)\,dA = \iint_D G(x,y,f(x,y))\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$$

and

$$\iint_S \mathbf{F}(x,y,z)\cdot\mathbf{n}\,dA = \iint_D \mathbf{F}(x,y,f(x,y))\cdot\left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right\rangle dx\,dy$$

Puzzler #3

Find the surface area of the part of the paraboloid $z = a^2 - x^2 - y^2$ lying above the xy-plane.



This surface is parametrized by

$$\mathbf{r}(x, y) = \langle x, y, a^2 - x^2 - y^2 \rangle.$$

What values of x and y are allowed?

Reminders for the week of November 13–17

- Exam #3 on Wednesday, November 15, 5:00 PM-7:00 PM
- No recitation on Thursday, November 16
- Webwork C7 on parametrized surfaces and tangent planes due Friday, November 17 by 11:59 PM
- Homework D1 on surface integrals due Monday, November 20