Math 213 - Gradient, Divergence, and Curl

Peter Perry

November 17, 2023

Unit D: Vector Calculus

- November 17 - Gradient, Divergence, Curl
- November 20 - The Divergence Theorem
- November 27 - Green's Theorem
- November 29 - Stokes' Theorem, Part I
- December 1 - Stokes' Theorem, Part II
- December 4 - Final Review
- December 6 - Final Review

Gradient, Divergence, Curl

Let $\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$

This lecture is about three vector derivatives:

- The *gradient* of a scalar function $f(x,y,z)$

$$(\nabla f)(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)\mathbf{i} + \frac{\partial f}{\partial y}(x,y,z)\mathbf{j} + \frac{\partial f}{\partial z}(x,y,z)\mathbf{j}$$

- The *divergence* of a vector field:

$$(\nabla \cdot \mathbf{F})(x,y,z) = \frac{\partial P}{\partial x}(x,y,z) + \frac{\partial Q}{\partial y}(x,y,z) + \frac{\partial R}{\partial z}(x,y,z)$$

- The *curl* of a vector field:

$$(\nabla \times \mathbf{F})(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y,z) & Q(x,y,z) & R(x,y,z) \end{vmatrix}$$

Derivatives of Vector Fields

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

Why choose the divergence and the curl out of all possible derivatives of a vector field?

$$\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{pmatrix}$$

Derivatives of Vector Fields

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

Why choose the divergence and the curl out of all possible derivatives of a vector field?

$$\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{pmatrix}$$

The *divergence* of a vector field,

$$(\nabla \cdot \mathbf{F})(x,y,z) = \frac{\partial P}{\partial x}(x,y,z) + \frac{\partial Q}{\partial y}(x,y,z) + \frac{\partial R}{\partial z}(x,y,z)$$

measures the flux of a vector field per unit volume

Derivatives of Vector Fields

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

Why choose the divergence and the curl out of all possible derivatives of a vector field?

$$\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} & \frac{\partial Q}{\partial z} \\ \frac{\partial R}{\partial x} & \frac{\partial R}{\partial y} & \frac{\partial R}{\partial z} \end{pmatrix}$$

The *curl* of a vector field,

$$(\nabla \times \mathbf{F}) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

measures the rotation (axis and speed) of a vector field at (x,y,z)

The Laplacian

If $\mathbf{A} = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$, then

$$\nabla \cdot \mathbf{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

If $f(x,y,z)$ is a scalar function, then

$$(\nabla f)(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Find

$$\nabla \cdot (\nabla f)$$

which is also denoted

$$\nabla^2 f$$

and called the *Laplacian* of f.

Notation Break

You may also see the notations

$$\nabla \times \mathbf{A} = \operatorname{curl} \mathbf{A}$$
$$\nabla \cdot \mathbf{A} = \operatorname{div} \mathbf{A}$$

for the curl and the divergence.

Several Thousand Identities and How to Guess Them

For functions f and g, vector fields $\mathbf{F}$ and $\mathbf{G}$, and a constant c,

$$\nabla(f+g) = \nabla f + \nabla g \qquad \nabla\cdot(\mathbf{F}+\mathbf{G}) = \nabla\cdot\mathbf{F} + \nabla\cdot\mathbf{G}$$

$$(cf) = c\nabla f \qquad \nabla\cdot(c\mathbf{F}) = c(\nabla\cdot\mathbf{F})$$

$$\nabla(fg) = (\nabla f)g + f(\nabla g) \qquad \nabla\cdot(f\mathbf{F}) = \nabla f\cdot\mathbf{F} + f\nabla\cdot\mathbf{F}$$

$$\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2} \qquad \nabla\cdot(\mathbf{F}\times\mathbf{G}) = (\nabla\times\mathbf{F})\cdot\mathbf{G} - \mathbf{F}\cdot(\nabla\times\mathbf{G})$$

$$\nabla\times(\nabla f) = 0 \qquad \text{curl of a gradient}$$

$$\nabla\cdot(\nabla\times\mathbf{F}) = \mathbf{0} \qquad \text{divergence of a curl}$$

To see lots of vector calculus identities, go to the relevant Wikipedia page!

Scalar and Vector Potentials

A scalar function φ is a *scalar potential* for a vector field $\mathbf{F}$ if

$$\mathbf{F} = \nabla\varphi$$

Screening test: if $\mathbf{F}$ is a gradient vector field then

$$\nabla \times \mathbf{F} = \mathbf{0}$$

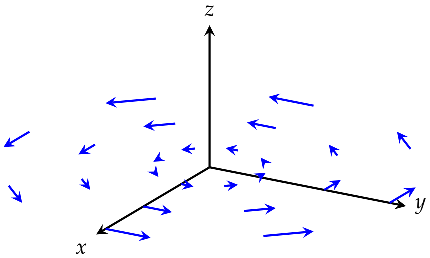
A vector field $\mathbf{A}$ is a *vector potential* for a vector field $\mathbf{B}$ if

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Screening test: If $\mathbf{B} = \nabla \times \mathbf{A}$ for a vector potential $\mathbf{A}$, then

$$\nabla \cdot \mathbf{B} = 0$$

Vector Potentials



Find a vector potential for the *solenoidal magentic field*

$$\mathbf{B} = -y\mathbf{i} + x\mathbf{j}$$

Remember that $\mathbf{B} = \nabla \times \mathbf{A}$ if

$$\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

Interpretation of the Gradient

The gradient of a scalar function is related to its change along a curve: if $f(x, y, z)$ is a function and

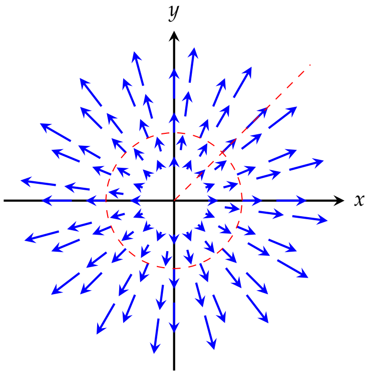
$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

then

$$\begin{aligned}\frac{d}{dt} f(x(t), y(t), z(t)) &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (\nabla f)(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\end{aligned}$$

The rate of change of f along the curve is *greatest* if $\mathbf{r}'(t)$ points in the direction of the gradient, and *least* if $\mathbf{r}'(t)$ is orthogonal to the gradient

Interpretation of the Gradient



Suppose $f(x,y) = x^2 + y^2$.

If $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$, find

$$\left.\frac{d}{dt} f(\mathbf{r}(t))\right|_{t=0}$$

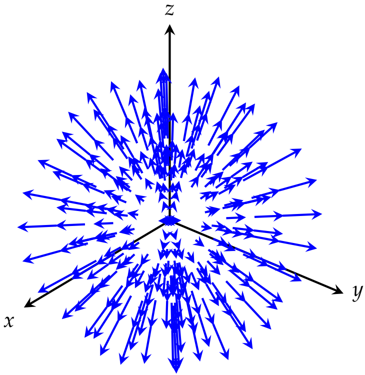
If $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}$, find

$$\left.\frac{d}{dt} f(\mathbf{r}(t))\right|_{t=1}$$

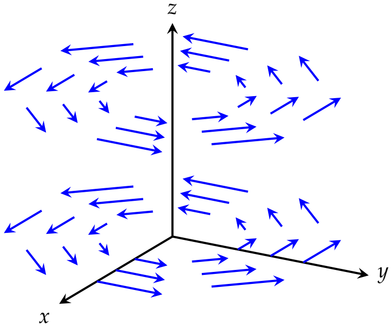
Interpretation of the Divergence

Let's look at two vector fields and their divergence.

$$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$\nabla \cdot \mathbf{F} = 3$$



$$\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$$
$$\nabla \cdot \mathbf{F} = 0$$



Interpretation of the Divergence

We will soon prove:

Divergence Theorem: If V is a bounded surface with piecewise smooth boundary ∂V, and $\mathbf{F}$ is a vector field with continuous first partial derivatives, then

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n}\, dS = \int_V \nabla \cdot \mathbf{F}\, dV$$

Now suppose that V is a sphere of radius ε centered at $\mathbf{r}_0$. Then

$$\int_{\partial V} \mathbf{F} \cdot \mathbf{n} dS = \text{flux of } \mathbf{F} \text{ across the boundary of the sphere}$$

$$\int_V \nabla \cdot \mathbf{F}\, dV = \text{volume integral of the divergence over the interior}$$

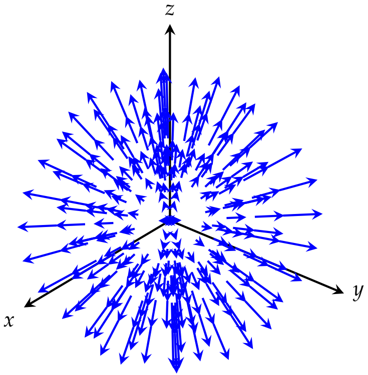
Conclusion: $(\nabla \cdot \mathbf{F})(\mathbf{r}_0)$ is the net flux of the vector field, per unit volume, per unit time

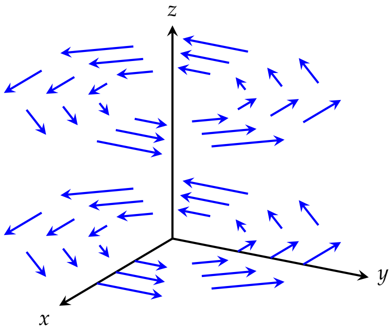
Interpretation of the Curl

Let's look at two vector fields and their curl:

$$\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$
$$\nabla \times \mathbf{F} = \mathbf{0}$$

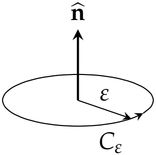
$$\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j}$$
$$\nabla \times \mathbf{F} = 2\mathbf{k}$$





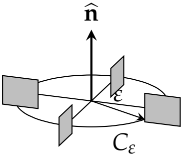
Interpretation of the Curl

If $\mathbf{r}_0$ is a point in $\mathbb{R}^3$, $\hat{\mathbf{n}}$ is a unit vector, and C_ε is a circle centered at $\mathbf{r}_0$, we claim that



$$(\nabla \times \mathbf{v})(\mathbf{r}_0) \cdot \hat{\mathbf{n}} = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi \varepsilon^2} \oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r}$$

The integral $\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r}$ is called the *circulation* of $\mathbf{v}$ around C_ε



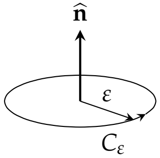
- If we parameterize C_ε by arc length,

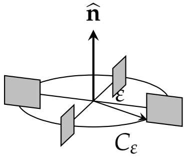
$$\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \int \mathbf{v}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{ds}\, ds$$

- If we visualize the fluid as moving a paddlewheel, the speed of the paddles, $\Omega\varepsilon$, should be the average value of $\mathbf{v}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{ds}$ around the circle

See CLP 4, 4.1.5 for more details

Interpretation of the Curl





- If we parameterize C_ε by arc length,

$$\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \int \mathbf{v}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{ds}\, ds$$

- If we visualize the fluid as moving a paddlewheel, the speed of the paddles, $\Omega\varepsilon$, should be the average value of $\mathbf{v}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{ds}$ around the circle
- The rate of rotation of the paddlewheels, Ω, should be determined by

$$\Omega\varepsilon = \frac{\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{ds}}{\oint_{C_\varepsilon} ds} = \frac{\oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r}}{2\pi\varepsilon}$$

So

$$\nabla \times \mathbf{v}(\mathbf{r}_0) \cdot \hat{\mathbf{n}} = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi\varepsilon^2} \oint_{C_\varepsilon} \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = 2\Omega$$

See CLP 4, 4.1.5 for more details

Reminders for the week of November 13–17 and November 20–24

- Webwork C7 on parametrized surfaces and tangent planes due Friday, November 17 by 11:59 PM
- Homework D1 on surface integrals due Monday, November 20
- Lecture on the Divergence Theorem, Monday November 20
- Thanksgiving Break, November 22-26