Math 213 - The Divergence Theorem

Peter Perry

November 20, 2023

Unit D: Vector Calculus

- November 17 - Gradient, Divergence, Curl
- November 20 - The Divergence Theorem
- November 27 - Green's Theorem
- November 29 - Stokes' Theorem, Part I
- December 1 - Stokes' Theorem, Part II
- December 4 - Final Review
- December 6 - Final Review

Fundamental Theorems

The Fundamental Theorem of Calculus states that

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

The Fundamental Theorem for Line Integrals of Conservative Vector Fields states that if $\mathcal{C}$ is any curve from P_0 to P_1, then

$$\int_{\mathcal{C}} (\nabla f) \cdot d\mathbf{r} = \varphi(P_1) - \varphi(P_0)$$

Both of these theorems take the form

$$\int_{\text{a path}} (\text{the derivative of } f) = f \text{ evaluated at the boundary}$$

For the next few weeks we will study new "fundamental theorems" which relate integrals over a *surface* or a *volume* to integrals over their *boundaries*

The Divergence Theorem

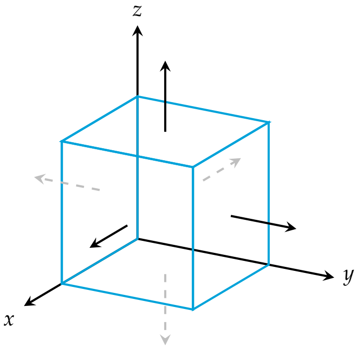
Theorem Suppose that $\mathbf{v}$ is a vector field with continuous first partial derivatives at every point in a bounded volume V with piecewise smooth boundary ∂V. Then

$$\iint_{\partial V} \mathbf{v} \cdot \widehat{\mathbf{n}}\, dS = \iiint_V (\nabla \cdot \mathbf{v})\, dV$$

where $\widehat{\mathbf{n}}$ is the outward unit normal to the surface ∂V.

This theorem is due to Carl Friedrich Gauss (German, 1777-1855) and Mikhail Ostrogradsky (Russian-Ukranian, 1801-1862)

The Divergence Theorem: Example



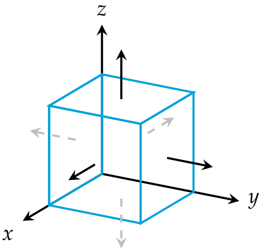
Suppose V is a cube of side a.

Then ∂V consists of six faces of the cube with outward normals as shown.

According to the divergence theorem, $\iiint_V (\nabla \cdot \mathbf{v})\, dV$ is the sum over the six faces $F_1, \ldots, F_6$ of the fluxes

$$\iint_{F_i} \mathbf{v} \cdot \widehat{\mathbf{n}}_i\, dS$$

The Divergence Theorem: Proof



One the one hand,

$$\iiint_V (\nabla \cdot \mathbf{v})\, dV =$$

$$\int_0^a \int_0^a \int_0^a \Bigg(\underbrace{\frac{\partial P}{\partial x}}_{(1)} + \underbrace{\frac{\partial Q}{\partial y}}_{(2)} + \underbrace{\frac{\partial R}{\partial z}}_{(3)} \Bigg)\, dx\, dy\, dz$$

On the other hand,

$$(1) = \int_0^a \int_0^a [P(a,y,z) - P(0,y,z)]\, dy\, dz = \text{flux through } x = a \text{ and } x = 0$$

$$(2) = \int_0^a \int_0^a [Q(x,a,y) - Q(x,0,y)]\, dx\, dz = \text{flux through } y = a \text{ and } y = 0$$

$$(3) = \int_0^a \int_0^a [R(x,y,a) - R(x,y,0)]\, dx\, dy = \text{flux through } z = a \text{ and } z = 0$$

Puzzler # 1

Suppose that

$$\mathbf{F}(x, y, z) = z\mathbf{k},$$

∂V is the sphere

$$x^2 + y^2 + z^2 = a^2,$$

and V is the interior of the sphere.

Find the flux of $\mathbf{F}$ out of V first by computing the surface integral $\iint_{\partial V} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$ (challenging!)

Hint: The sphere is parameterized by

$$\mathbf{r}(\theta, \varphi) = (a \sin\varphi \cos\theta)\mathbf{i} + (a \sin\varphi \sin\theta)\mathbf{j} + (a \cos\varphi)\mathbf{k}$$

and

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = (a^2 \sin\varphi) \frac{\mathbf{r}}{|\mathbf{r}|}$$

Puzzler #1, Continued

$$\iint_{\partial V} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV$$

$$\mathbf{F}(x, y, z) = z\mathbf{k}$$

V is the sphere of radius a centered at the origin

Now compute the flux of **F** out of the sphere by computing the volume integral $\iiint_V (\nabla \cdot \mathbf{F})\, dV$ (easy!).

Puzzler #2

$$\iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iiint_V (\nabla \cdot \mathbf{F}) \, dV$$

Use the divergence theorem to calculate the flux of **F** across S if

$$\mathbf{F}(x, y, z) = z\mathbf{i} + y\mathbf{j} + zx\mathbf{k}$$

and S is the surface of the tetrahedron enclosed by the coordinate planes at the plane

$$\frac{x}{3} + \frac{y}{5} + \frac{z}{5} = 1$$

Puzzler #3

$$\iint_{\partial V} \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV$$

Let

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and let V be the surface bounded by the ellipsoid

$$x^2 + \frac{y^2}{4} + \frac{z^2}{4} = 1$$

Find the flux of **F** out of ∂V. It may be useful to know that the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has volume $\frac{4}{3}\pi abc$

A Cautionary Tale (Part I)

Suppose that

$$\mathbf{F}(x,y,z) = \frac{1}{|\mathbf{r}|^3}\mathbf{r}$$

where

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If V is the unit sphere centered at $(0,0,0)$, let's find the flux of $\mathbf{F}$ out of ∂V.

Note that for the unit sphere

$$\int_{\partial V} \mathbf{F}\cdot\widehat{\mathbf{n}}\,dS = \int_{|\mathbf{r}|=1} \mathbf{F}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}\,dS$$

A Cautionary Tale (Part II)

$$\iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iiint_V (\nabla \cdot \mathbf{F}) \, dV$$

For the same vector field

$$\mathbf{F}(x, y, z) = \frac{1}{|\mathbf{r}|^3} \mathbf{r}$$

what is $\nabla \cdot \mathbf{F}$?

Write $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{3}{2}} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and compute

$$\frac{\partial P}{\partial x} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} - 3x^2 (x^2 + y^2 + z^2)^{-\frac{5}{2}}$$
$$\frac{\partial Q}{\partial y} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} - 3y^2 (x^2 + y^2 + z^2)^{-\frac{5}{2}}$$
$$\frac{\partial R}{\partial z} = (x^2 + y^2 + z^2)^{-\frac{3}{2}} - 3z^2 (x^2 + y^2 + z^2)^{-\frac{5}{2}}$$

and notice the the three components sum to zero, so $\nabla \cdot \mathbf{F} = 0$.

Variations on the Divergence Theorem

There are several versions of the Divergence Theorem other than the one we've presented today.

If V is a solid with surface ∂V, then:

$$\iint_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}} \, dS = \iiint_V (\nabla \cdot \mathbf{F}) \, dV$$

$$\iint_{\partial V} f \hat{\mathbf{n}} \, dS = \iiint_V (\nabla f) \, dV$$

$$\iint_{\partial V} \hat{\mathbf{n}} \times \mathbf{F} \, dS = \iiint_V \nabla \times \mathbf{F} \, dV$$

For a better sense of the full sweep of vector calculus, see for example Wikipedia's Vector Calculus Page–scroll down to the secion on Integration

Reminders for the week of November 20–24 and November 27–December 1

- Homework D1 on surface integrals due Monday, November 20
- Thanksgiving Break, November 22-26
- Homework D2 on Gradient, Divergence, and Curl is due on Wednesday, November 29
- Homework D3 on the Divergence Theorem is due on Friday, December 1