Math 213 - Green's Theorem

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Unit D: Vector Calculus

- November 17 - Gradient, Divergence, Curl
- November 20 - The Divergence Theorem
- November 27 - Green's Theorem
- November 29 - Stokes' Theorem, Part I
- December 1 - Stokes' Theorem, Part II
- December 4 - Final Review
- December 6 - Final Review

Preview

Green's Theorem relates the line integral of a vector field around a oriented, simple, closed curve C to an integral of its derivative over the enclosed surface R.
The statement is:

$$\oint_C F(x,y)\,dx + G(x,y)\,dy = \iint_R \left(\frac{\partial G}{\partial x}(x,y) - \frac{\partial F}{\partial y} \right) dx\,dy$$

To understand and use Green's theorem, we need to:

1. Review line integrals of vector fields
2. Define what a simple closed curve is
3. Define the *orientation* of a curve means, and what is the *right* orientation for curves bounding the enclosed region

(Vector) Line Integrals

If C is a parametrized curve

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, \quad a \leq t \leq b$$

then

$$dx = x'(t)\,dt, \quad dy = y'(t)\,dt$$

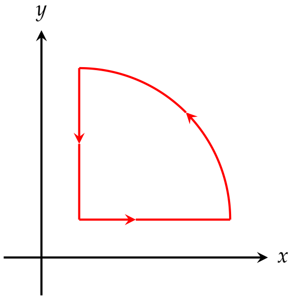
and

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \left[P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\right]\,dt$$

Express $\int_C x^2y\,dx + xy\,dy$ as an integral with respect to t if C is the curve

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}.$$

Vocabulary - Simple, Closed, Piecewise Smooth



Suppose that $\mathcal{C}$ is a parameterized curve with parametrization $\mathbf{r}(t)$, $a \leq t \leq b$.

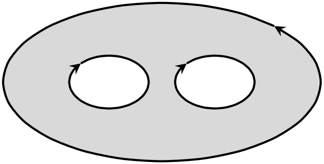
$\mathcal{C}$ is *closed* if $\mathbf{r}(a) = \mathbf{r}(b)$.

$\mathcal{C}$ is *simple* if it does not cross itself.

$\mathcal{C}$ is *piecewise smooth* if $\mathcal{C}$ has a parametrization that is continuous and differentiable except at finitely many points

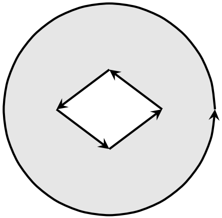
Which of these is closed, simple, piecewise smooth?

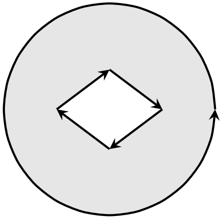
Vocabulary - Oriented Boundary



If R is a region in the xy plane bounded by finitely many closed, simple, piecewise smooth curves, the *oriented* boundary (or, the boundary with *positive orientation*) gives the bounding curves a direction so that R is always to the *left* as you walk along the curve

Which of these two boundaries has the correct (positive) orientation?





Why Orientation Matters

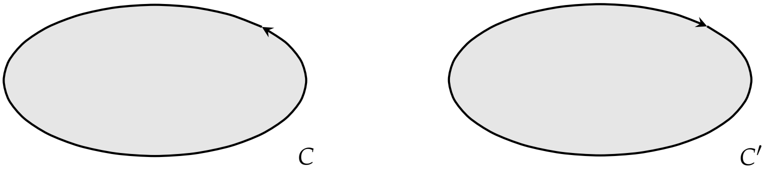
What is the relationship between

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy$$

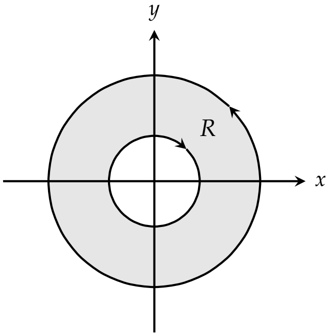
and

$$\oint_{C'} P(x,y)\,dx + Q(x,y)\,dy$$

if C and C' are the oriented curves shown below?



Green's Theorem



Suppose R is a region in the xy-plane whose oriented boundary C consists of finitely many piecewise smooth curves.

Suppose F and G have continuous first partials at every point of R.

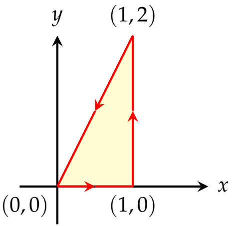
Then

$$\oint_C F(x,y)\,dx + G(x,y)\,dy = \iint_R \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx\,dy.$$

Notice that the boundary curves have *positive orientation*

Puzzler #1

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$



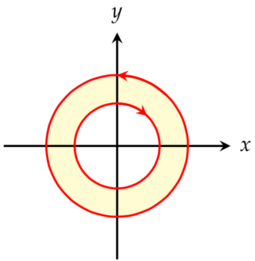
Use Green's Theorem to evaluate

$$\oint_C xy\,dx + x^3y^3\,y$$

if C is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Puzzler #2

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy.$$



Use Green's Theorem to evaluate

$$\oint_C (7x+y^2)dy - (x^2-2y)dx$$

if C is the oriented curve shown at left.

Green's Theorem and Areas

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dx\,dy.$$

Corollary Suppose R is a region in the xy plane with piecewise smooth oriented boundary C. Then

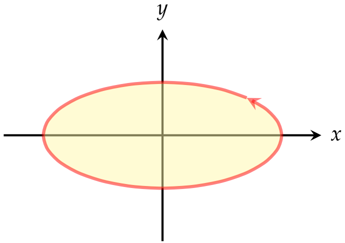
$$\text{Area}(R) = \oint_C x\,dy = -\oint y\,dx = -\frac{1}{2}\oint_C x\,dy - y\,dx$$

Proof: Apply Green's Theorem to the vector fields

$$\mathbf{F}_1 = x\mathbf{i}, \quad \mathbf{F}_2 = y\mathbf{j}, \quad \mathbf{F}_3 = \frac{1}{2}(x\mathbf{i} - y\mathbf{j})$$

Puzzler #3

$$\text{Area}(R) = \oint_C x\,dy = -\oint y\,dx = -\frac{1}{2}\oint_C x\,dy - y\,dx$$



Find the area of the region bounded by the ellipse

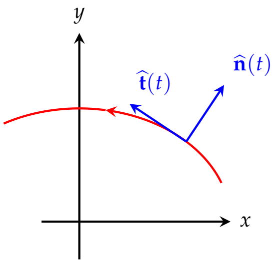
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The ellipse is parametrized by

$$\mathbf{r}(t) = a\cos(t)\mathbf{i} + b\sin(t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

Unit Tangent and Unit Normal

If C is a curve parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$:



The *unit tangent* is

$$\widehat{\mathbf{t}}(t) = \frac{1}{|\mathbf{r}'(t)|}\left(x'(t)\mathbf{i} + y'(t)\mathbf{j}\right)$$

The *unit normal* is

$$\widehat{\mathbf{n}}(t) = \frac{1}{|\mathbf{r}'(t)|}\left(y'(t)\mathbf{i} - x'(t)\mathbf{j}\right)$$

For later use note that

$$\widehat{\mathbf{n}}(t)\,ds = \frac{1}{|\mathbf{r}'(t)|}\left(y'(t)\mathbf{i} - x'(t)\mathbf{j}\right)\,|\mathbf{r}'(t)|\,dt$$

Another Form of Green's Theorem

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

If R is a region in the xy plane whose boundary consists of a single simple, closed curve, then

$$\iint_R \nabla \cdot \mathbf{F}\,dx\,dy = \oint_C \mathbf{F} \cdot \widehat{\mathbf{n}}\,ds$$

where $\widehat{\mathbf{n}}$ is the unit normal to the curve C.

Proof: Apply Green's Theorem above to

$$P(x,y) = -G(x,y), \quad Q(x,y) = F(x,y)$$

and remember that

$$\widehat{\mathbf{n}}(t) = \frac{1}{|\mathbf{r}'(t)|}\left(y'(t)\mathbf{i} - x'(t)\mathbf{j}\right)$$

Reminders for the week of November 27–December 1

- Homework D2 on Gradient, Divergence, and Curl is due on Wednesday, November 29
- Homework D3 on the Divergence Theorem is due on Friday, December 1