Math 213 - Cross Products

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August 28, 2023

Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

Determinants

2×2 Determinants

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

3×3 Determinants

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinant Practice

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Find the following determinants:

$$\begin{vmatrix} 2 & 1 \\ 4 & -6 \end{vmatrix} = (2)(-6) - (1)(4) = -16$$

$$\begin{vmatrix} 4 & -6 \\ 2 & 1 \end{vmatrix} = (4)(1) - (-6)(2) = 16.$$ Note rows are reversed from example above

$$\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1)(4) - (2)(2) = 0.$$ Note second row is a multiple of the first

Determinant Practice

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Find:

$$\begin{vmatrix} 1 & -4 & -2 \\ 2 & 0 & 4 \\ -1 & 2 & 3 \end{vmatrix} = 1\begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix} + 4\begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} + (-2)\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix} = 1 \cdot (-8) + 4 \cdot (10) - 2 \cdot (4)$$

$$= 24$$

using your favorite method

The Cross Product of Two Vectors

The cross product of

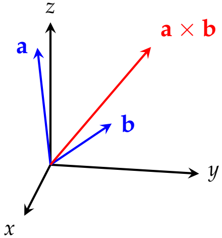
$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

is a new *vector* given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The Cross Product of Two Vectors

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



Find $\mathbf{a} \times \mathbf{b}$ if $\mathbf{a} = \langle 1, 0, 2 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$

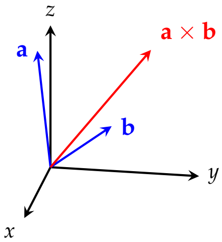
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$

The Cross Product of Two Vectors

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a} = \mathbf{i} + 0\mathbf{j} + 2\mathbf{k}$$
$$\mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$
$$\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$$



What are $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b})$?

You can check that both dot products are zero, which shows that $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$

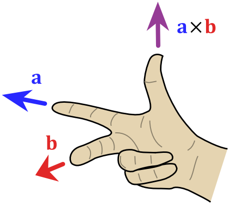
What is $\mathbf{b} \times \mathbf{a}$?

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 2\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Cross Product Properties

1. The cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ with direction given by the right-hand rule:



Source: Wikipedia Commons

2. $\mathbf{b} \times \mathbf{a} = -(\mathbf{a} \times \mathbf{b})$
3. $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, i.e.,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$$

4. Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

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Find $\mathbf{i} \times \mathbf{j}$, $\mathbf{i} \times \mathbf{k}$, and $\mathbf{j} \times \mathbf{k}$

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$

Find $(2\mathbf{j} - 4\mathbf{k}) \times (-\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

$14\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

Cross Product Properties

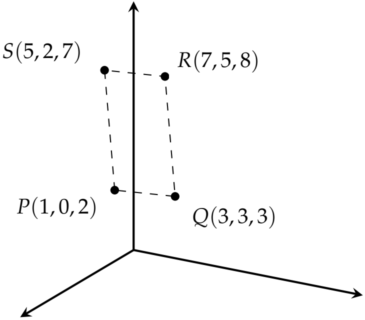
Can you fill in the blanks?

If **a**, **b**, and **c** are vectors and if c is a scalar:

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = \_(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\_\_)$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \_\_\_\_\_\_\_\_$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \_\_\_\_\_\_\_\_$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

1.
2. $(c\mathbf{a}) \times \mathbf{b} = \underline{c}(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\underline{c\mathbf{b}})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \underline{\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \underline{\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}}$
5.
6.

Cross Product Puzzler #1



Find the area of the parallelogram with vertices

$$P(1,0,2), \quad Q(3,3,3), (7,5,8), \quad S(5,2,7).$$

Note that:

$$\overrightarrow{PQ} = \langle 2,3,1 \rangle, \quad \overrightarrow{PS} = \langle 4,2,5 \rangle$$

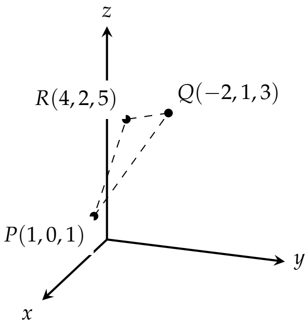
The area of the parallelogram is the length of $\overrightarrow{PQ} \times \overrightarrow{PS}$. We can compute

$$\overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{vmatrix} = 13\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}$$

and

$$|\overrightarrow{PQ} \times \overrightarrow{PS}| = \sqrt{13^2 + 6^2 + 8^2} = \sqrt{269}$$

Cross Product Puzzler #2

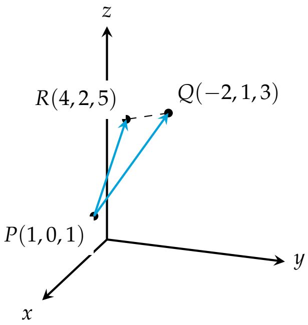


Find a nonzero vector orthogonal to the plane through

$$P(1,0,1),\ Q(-2,1,3),\ R(4,2,5)$$

and find the area of triangle PQR

Cross Product Puzzler #2



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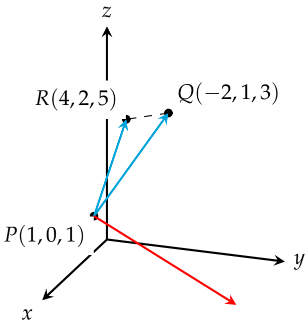
Two sides of the triangle are spanned by:

$$\overrightarrow{PR} = \langle 3,2,4 \rangle, \quad \overrightarrow{PQ} = \langle -3,1,2 \rangle$$

Cross Product Puzzler #2



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Two sides of the triangle are spanned by:

$$\overrightarrow{PR} = \langle 3,2,4\rangle, \quad \overrightarrow{PQ} = \langle -3,1,2\rangle$$

An orthogonal vector is $\overrightarrow{PR} \times \overrightarrow{PQ}$, and the length of this vector is twice the area of the triangle (why?). You can compute that

$$\overrightarrow{PR} \times \overrightarrow{PQ} = -18\mathbf{j} + 9\mathbf{k}$$

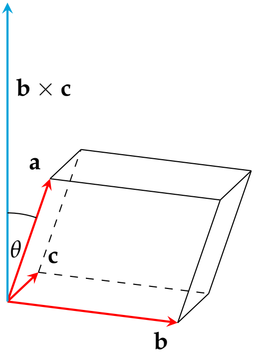
and

$$|\overrightarrow{PR} \times \overrightarrow{PQ}| = 9\sqrt{5}$$

The Scalar Triple Product

The *scalar triple product* of three vectors **a**, **b**, **c** is the determinant

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

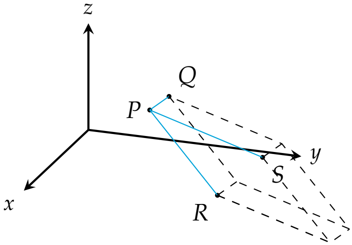


The volume of the parallelepiped formed by the vectors **a**, **b**, and **c** is given by

$$|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

What happens if the vectors **a** , **b**, and **c** are coplanar?

Triple Product Puzzler



Find the volume of the parallelepiped with adjacent ages PQ, PR, and PS if

$$P = P(-2,1,0), \quad Q = (2,3,2),$$
$$R = R(1,4,-1), \quad S = S(3,6,1)$$

$$\overrightarrow{PQ} = \langle 4,2,2 \rangle$$

$$\overrightarrow{PR} = \langle 3,3,-1 \rangle$$

$$\overrightarrow{PS} = \langle 5,5,1 \rangle$$

The volume is $\begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} = 16$

Dot Product, Cross Product, Triple Product

	Dot Product	Cross Product	Scalar Triple Product
Type	Scalar $\mathbf{a} \cdot \mathbf{b}$	Vector $\mathbf{a} \times \mathbf{b}$	Scalar $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
Magnitude	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \sin\theta$	
Symmetry	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	Antisymmetric
Direction	None!	Right-hand rule	None!
In Physics	Work	Torque	
In Geometry	Projection	Parallelogram Area	Parallelepiped Area
Zero If	$\mathbf{a} \perp \mathbf{b}$	$\mathbf{a} \parallel \mathbf{b}$	$\mathbf{a}, \mathbf{b}, \mathbf{c}$ coplanar

Reminders for the Week of August 28-September 1

- Recitation on dot products, cross products Tuesday 8/29
- Read CLP 3, section 1.4, Equations of Planes for Wednesday 8/30
- WebWork A2 due on Wednesday 8/30 by 11:59 PM
- Recitation on planes Thursday 8/31
- Quiz # 1 on coordinate systems and vectors due on Thursday 8/31 at 11:59 PM
- Read CLP 3, section 1.5 for Friday 9/1