Math 213 - Stokes' Theorem

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November 29, 2023

Unit D: Vector Calculus

- November 17 - Gradient, Divergence, Curl
- November 20 - The Divergence Theorem
- November 27 - Green's Theorem
- November 29 - Stokes' Theorem, Part I
- December 1 - Stokes' Theorem, Part II
- December 4 - Final Review
- December 6 - Final Review

Preview

Stokes' Theorem generalizes Green's Theorem to three dimensions.

Green's Theorem states that if R is a plane region with simple, piecewise smooth, closed boundary C, then

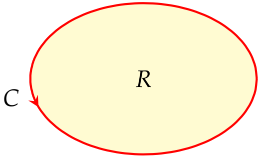
$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Stokes' Theorem states that if $\mathbf{F}$ is a vector field and S is an oriented surface with simple, piecewise smooth boundary C, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\,dS$$

where $\widehat{\mathbf{n}}$ is a unit normal to S oriented consistently with C.

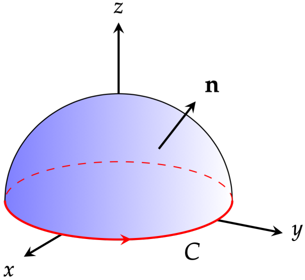
Green versus Stokes



$$\oint_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA$$

Consistency:

If you walk along C in the direction of the arrow, the region is to the left



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times F \cdot \hat{\mathbf{n}}\,dS$$

Consistency:

If you walk along C in the direction of the arrow with the vector from your feet to your head having direction $\hat{\mathbf{n}}$, then S is on your left hand side

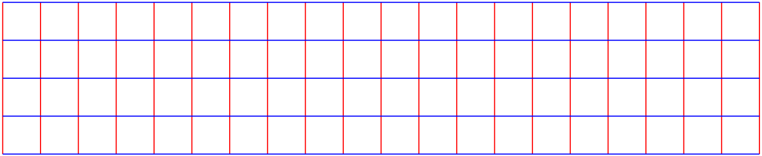
Oriented Surfaces

An *oriented surface* is a surface with a unit normal for each point on the surface that depends continuously on the point

An (in)famous example of a non-orientable surface is the Möbius strip (there's an app for that!)

If a surface has two sides, you can orient the surface by choosing one side to be the positive side. However, the Möbius strip has only one side!

Begin with a strip:



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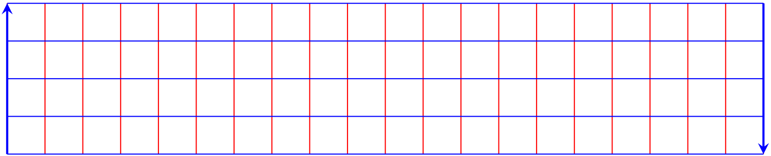


Twist the right-hand side and connect to the left to get a one-sided figure

Stokes' Theorem

Sir George Gabriel Stokes (1819-1903), Lucasian Professor of Mathematics at Cambridge University

Theorem

Suppose that S is an oriented smooth surface (a unit normal $\widehat{\mathbf{n}}$ is chosen at each point and varies continuously) whose boundary C consists of a finite number of piecewise smooth curves oriented consistently with $\widehat{\mathbf{n}}$.

Suppose that $\mathbf{F}$ is a vector field with continuous first partial derivatives at every point of S.

Then

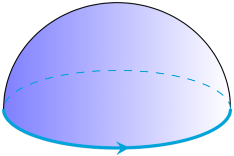
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

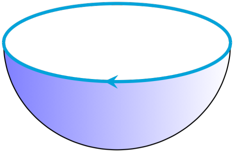
Orientation – Pick A Side

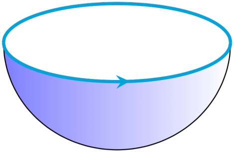
For Stokes' Theorem, the unit normal $\hat{\mathbf{n}}$ should obey the following consistency condition:

If you along the boundary C in the direction of the arrow with the vector from your feet to your head having direction $\hat{\mathbf{n}}$, then the surface S is on your left hand side

What is the correct direction for the normal in each of the following examples?





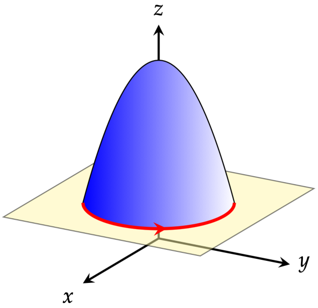


Using Stokes' Theorem: Example 1

Let S be the part of the surface $z = 5 - x^2 - y^2$ above the plane $z = 1$ and let

$$\mathbf{F}(x, y, z) = z^2\mathbf{i} - 3xy\mathbf{j} + x^3y^3\mathbf{k}.$$

Find $\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$.



According to Stokes' Theorem, we can instead compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the oriented boundary of S

What is C?

How can we (correctly) parametrize C?

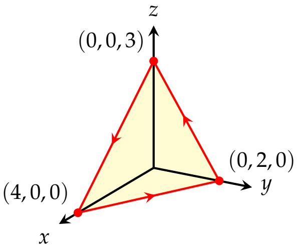
What is $\mathbf{F}$ along C?

Example courtesy of Paul's Online Math Notes, Examples for §17.5

Using Stokes Theorem: Example 2

Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ if C is the triangle with vertices $(0,0,3)$, $(0,2,0)$, and $(4,0,0)$ with orientation shown, and

$$\mathbf{F}(x,y,z) = (3yx^2 + z^3)\mathbf{i} + y^2\mathbf{j} + 4yx^2\mathbf{k}$$



Here we'll use Stokes' Theorem in the other direction

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

What is S?

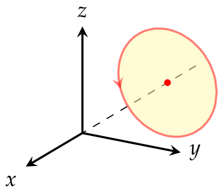
How do we parameterize S?

What is the correct outward normal for S?

Example courtesy of Paul's Online Math Notes, Examples for §17.5

Using Stokes' Theorem: Example 3

Use Stokes' Theorem to find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = x^2\mathbf{i} - 4z\mathbf{j} + xy\mathbf{k}$ and C is the circle of radius 1 at $z = -3$ and perpendicular to the x-axis with the orientation shown.



According to Stokes' Theorem, we can instead compute

$$\iint_S \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

if we can find an oriented surface S bounded by C.

What is the surface S?

How do we parametrize it?

Which way should the unit normal point?

Example courtesy of Paul's Online Math Notes, Problems for §17.5

Reminders for the week of November 27–December 1

- Homework D2 on Gradient, Divergence, and Curl is due on Wednesday, November 29
- Homework D3 on the Divergence Theorem is due on Friday, December 1