Math 213 - Stokes' Theorem

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Unit D: Vector Calculus

- November 17 - Gradient, Divergence, Curl
- November 20 - The Divergence Theorem
- November 27 - Green's Theorem
- November 29 - Stokes' Theorem, Part I
- December 1 - Stokes' Theorem, Part II
- December 4 - Final Review
- December 6 - Final Review

Stokes' Theorem

Theorem

Suppose that S is an oriented smooth surface (a unit normal $\widehat{\mathbf{n}}$ is chosen at each point and varies continuously) whose boundary C consists of a finite number of piecewise smooth curves oriented consistently with $\widehat{\mathbf{n}}$.

Suppose that $\mathbf{F}$ is a vector field with continuous first partial derivatives at every point of S.

Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

Green Versus Stokes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

Suppose that S is a domain in the xy plane with boundary C, and

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$

Then

$$\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

and

$$\widehat{\mathbf{n}} = \mathbf{k}$$

so we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

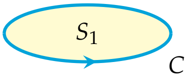
which is Green's Theorem

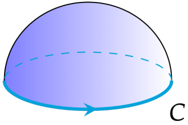
Stokes Versus Stokes

Suppose that S_1 and S_2 are oriented surfaces with the same oriented boundary C, and $\mathbf{F}$ is a vector field. Then

$$\int_{S_1} \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \int_{S_2} \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

Why is this true?





What is $\int_{S_2} \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS - \int_{S_1} \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$?

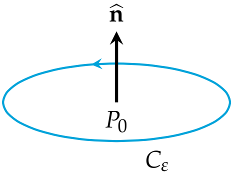
Remember that for any volume V bounded by a closed surface S,

$$\iint_{\partial V} \mathbf{G} \cdot \widehat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{G}\, dV$$

(the divergence theorem) and that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field $\mathbf{F}$.

Stokes Versus Curl

Suppose $\mathbf{v}$ is the velocity field of a fluid. Suppose D_ε is a disc of size ε centered at P_0 with unit normal $\widehat{\mathbf{n}}$, and let C_ε be its boundary.



By Stokes' Theorem,

$$\int_{D_\varepsilon} \nabla \times \mathbf{v} \cdot \widehat{\mathbf{n}}\, dS = \oint_{C_\varepsilon} \mathbf{v} \cdot d\mathbf{r}$$

or

$$\pi\varepsilon^2 (\nabla \times \mathbf{v})(P_0) \cdot \widehat{\mathbf{n}} \simeq \oint_{C_\varepsilon} \mathbf{v} \cdot d\mathbf{r}$$

The right-hand side is the *circulation* of the vector fiedl $\mathbf{v}$ around C_ε, and is approximately

$$2\pi\varepsilon v_T = 2\pi\varepsilon(\varepsilon\Omega)$$

where v_T is the average tangential velocity and Ω is the angular velocity in the $\widehat{\mathbf{n}}$ direction. So

$$\pi\varepsilon^2 (\nabla \times \mathbf{v})(P_0) \simeq 2\pi\varepsilon^2\Omega$$

$$\frac{1}{2}(\nabla \times \mathbf{v})(P_0) \cdot \widehat{\mathbf{n}} = \Omega$$

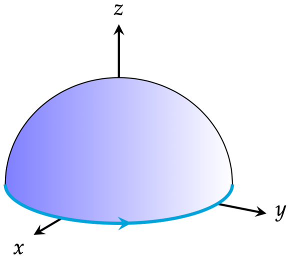
Puzzler # 1

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} \, dS$$

Find $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} \, dS$ if $\mathbf{F} = (z - y)\mathbf{i} + x\mathbf{j} - x\mathbf{k}$ and S is the hemisphere

$$x^2 + y^2 + z^2 = 4, \quad z \geq 0$$

oriented so that surface normals point away from the center of the sphere.



What is C, the boundary of S?

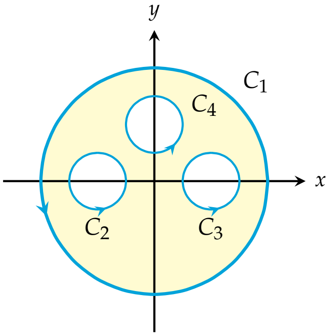
How do we parametrize it with the correct orientation?

What is $\mathbf{F}$ restricted to C?

Puzzler #2

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}}\, dS$$

A vector field $\mathbf{F}$ has $(\nabla \times \mathbf{F}) \cdot \widehat{\mathbf{n}} = 10$ inside the domain shown, where C_1 has radius 5 and the smaller circles C_2, C_3 and C_4 have radius 1. If $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi$ and $\oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = 2\pi$, what is $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$?



What does this problem have to do with Stokes' Theorem?

What is $\iint (\nabla \times \mathbf{F}) \cdot \widehat{\mathbf{n}}\, dS$ over the shaded region?

What does Stokes' Theorem say about the relation between $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and the other line integrals?

Puzzler #3

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \widehat{\mathbf{n}} \, dS$$

Suppose that $\mathbf{F}$ is a conservative vector field, i.e., $\mathbf{F} = \nabla \varphi$ for a scalar function φ. What does Stokes' Theorem allow to conclude about $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for any simple closed curve C? It may help to remember that, for any function φ,

$$\nabla \times (\nabla \varphi) = \mathbf{0}.$$

Three Kinds of Differentiation, Part I

The *gradient* of a function φ is a vector field

$$\nabla\varphi(x,y,z) = \left\langle \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right\rangle$$

which points in the direction of greatest change of φ. If $\mathbf{F} = \nabla\varphi$ for some potential function φ, then $\mathbf{F}$ is a *conservative* vector field

The *divergence* of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a scalar function

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

which measures the next flux of the vector field per unit area per unit time out of a small ball.

Three Kinds of Differentiation, Part II

The *curl* of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a new vector field

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

which measures the magnitude and direction of angular velocity if $\mathbf{F}$ is interpreted as a velocity field.

Scalar and Vector Potentials

A vector field $\mathbf{F}$ is called *irrotational* (new word!) if $\nabla \times \mathbf{F} = \mathbf{0}$. In this case, $\mathbf{F} = \nabla \varphi$ for a scalar function φ called the potential of $\mathbf{F}$. The condition $\nabla \times \mathbf{F} = \mathbf{0}$ means that there is no infinitesimal rotation at any point.

A vector field $\mathbf{B}$ is called *solenoidal* (new word!) if $\nabla \cdot \mathbf{B} = 0$. In this case $\mathbf{B} = \nabla \times \mathbf{A}$ for a vector field $\mathbf{A}$ called the *vector potential* for $\mathbf{B}$. The condition $\nabla \cdot \mathbf{B} = 0$ means that there are no sources or sinks.

Three Big Theorems, Part I

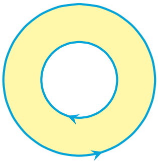
The Divergence Theorem If V is a bounded volume with piecewise smooth boundary S, and $\mathbf{F}$ is a vector field with continuous partial derivatives throughout V, then

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV$$

Green's Theorem If S is a bounded domain in the xy plane with piecewise smooth boundary C, and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field with continuous first partial derivatives throughout S, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where C is oriented so that S is always "to the left"

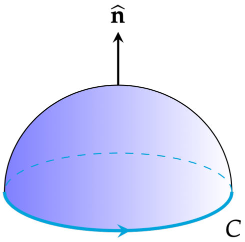


Three Big Theorems, Part II

Stokes' Theorem If S is a bounded, oriented surface with piecewise smooth boundary C,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

where $\mathbf{n}$ is chosen so that the path C is traversed with surface to the left by a walker standing in the direction of $\mathbf{n}$



Reminders for the week of November 27–December 1

- Homework D3 on the Divergence Theorem is due today, Friday, December 1