Math 213 - Semester Review, Part I

Peter Perry

December 4, 2023

Unit D: Vector Calculus

- November 17 - Gradient, Divergence, Curl
- November 20 - The Divergence Theorem
- November 27 - Green's Theorem
- November 29 - Stokes' Theorem, Part I
- December 1 - Stokes' Theorem, Part II
- December 4 - Final Review
- December 6 - Final Review

Final Exam Reminders

Your final exam takes place on Thursday, December 14, 6:00 PM-8:00 PM

Section 011	CP 139
Sections 012-014	CP 153

The final exam will consist of:

- 10 multiple choice questions covering Units A-D
- 4 free response questions focussing on Unit D together with material on conservative vector fields, parametrized surfaces, and surface integrals from from Unit C.

You are allowed one sheet of notes on notebook-sized paper. Notes on both sides are OK.

The Big Picture

What was this course about?

Unit A Three-dimensional space, vector algebra, equations of lines and planes, space curves, quadric surfaces

Unit B Differential calculus for functions of several variables: partial derivatives, chain rule, tangent planes and normal lines, the gradient, maxima and minima, Lagrange multiplier method

Unit C Integral calculus for functions of several variables: double integrals in Cartesian and polar coordinates, triple integrals in Cartesian, cylindrical, spherical coordinates, general coordinate changes. "Preview" topics: vector fields, line integrals, conservative vector fields. Integrals of scalar functions and vector fields over parametrized surfaces.

Unit D Vector Calculus: gradient, divergence, and curl. "Fundamental theorems" of vector calculus: the divergence theorem, Green's theorem, and Stokes' Theorem.

Vector Algebra

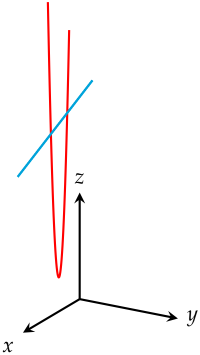
Can you fill in the following facts about basic vector operations?

Operation	**Meaning**	**Uses**
Dot product $\mathbf{a} \cdot \mathbf{b}$	$\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$	Test for . . .
		Define projection
Cross product $\mathbf{a} \times \mathbf{b}$	$\lvert\mathbf{a} \times \mathbf{b}\rvert = \lvert\mathbf{a}\rvert\,\lvert\mathbf{b}\rvert\,\sin\theta$	Test for . . .
		Area of . . .
Triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	Signed volume of . . .	

How do you compute each?

Curves

Do the following curves collide? Do they intersect?



$$\mathbf{r}(s) = 2s\mathbf{i} + (s-1)\mathbf{j} + (s^2+1)\mathbf{k}$$
$$\mathbf{w}(t) = (t^2+2)\mathbf{i} + (t^2-2)\mathbf{j} + (t^2+6)\mathbf{k}$$

Tangent Planes, Normal Lines, Linear Approximation

1. Find the equation of the tangent plane to the graph of $z = e^{x-y}$ at $(2,2,1)$.

$$z = 1 + x - y$$

2. Find the linear approximation to the function $f(x,y) = (y-1)/(x+1)$ at $(0,0)$.

$$f_x(x,y) = -\frac{y-1}{(x+1)^2} \qquad f_y(x,y) = \frac{1}{x+1}$$

so

$$L(x,y) = L(0,0) + f_x(0,0)x + f_y(0,0)y = -1 + x + y$$

3. Find an equation of the normal line to $z = 2x^2 - y^2 - 5y$ at $(1,2,-4)$.
Find the gradient of $f(x,y,z) = z - 2x^2 + y^2 + 5y$ at $(1,2,-4)$:

$$\nabla f(x,y,z) = \langle -4x, 2y+5, 1 \rangle$$

$$\nabla f(1,2,-4) = \langle -4, 9, 1 \rangle$$

so

$$\mathbf{r}(t) = \langle 1-4t, 2+9t, -4+t \rangle$$

Differential Calculus - Chain Rule

What is the chain rule tree for $f(x, y)$ if x and y are functions of u, v, w? How do you compute $\frac{\partial f}{\partial v}$?

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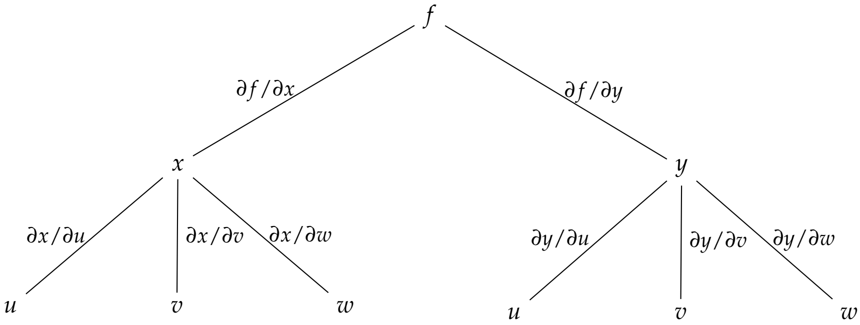


Differential Calculus - Gradient

Recall that

$$(\nabla f)(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)\mathbf{i} + \frac{\partial f}{\partial y}(x,y,z)\mathbf{j} + \frac{\partial f}{\partial z}(x,y,z)\mathbf{k}$$

In which direction does $(\nabla f)(x,y,z)$ point?

What is the magnitude of $(\nabla f)(x,y,z)$?

If $\mathbf{u}$ is a unit vector, how can you use the gradient to find $D_{\mathbf{u}}f(a,b,c)$?

If $f(x,y,z)$ has a local maximum or local minimum at (a,b,c), what is $(\nabla f)(a,b,c)$?

Differential Calculus - Maxima and Minima

Suppose that $f(x,y)$ is a function of two variables and that

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

How can you tell whether (a,b) is a local maximum, a local minimum, or a saddle point?

$$D = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(a,b) & \frac{\partial^2 f}{\partial x \partial y}(a,b) \\ \frac{\partial^2 f}{\partial y \partial x}(a,b) & \frac{\partial^2 f}{\partial y^2}(a,b) \end{vmatrix}$$

Type	$f_{xx}(a,b)$	D
Maximum	< 0	> 0
Minimum	> 0	> 0
Saddle	$\ldots$	< 0

Differential Calculus - Maxima and Minima

Find the critical points of the function

$$f(x,y) = (y-2)x^2 - y^2$$

and determine whether they are local maxima, local minima, or neither.

$$f_x(x,y) = 2x(y-2), \quad f_y(x,y) = x^2 - 2y$$

so critical points occur at $(0,0)$, $(2,2)$ and $(-2,2)$. Next,

$$D(x,y) = \begin{vmatrix} 2(y-2) & 2x \\ 2x & -2 \end{vmatrix} = 8 - 4y - 4x^2$$

x	y	$f_{xx}(x,y)$	$D(x,y)$	Type
0	0	-4	8	Local Maximum
2	2	0	-16	Saddle
-2	2	0	-16	Saddle

Source: Paul's Online Math Notes, §14.3, Example 1

Differential Calculus - Lagrange Method

To find extrema of a function

$$f(x,y) \quad \text{Objective Function}$$

subject to

$$g(x,y) = 0 \quad \text{Constraint Function}$$

solve the Lagrange equations

$$(\nabla f)(x,y) = \lambda(\nabla g)(x,y)$$
$$g(x,y) = 0$$

for (x,y) and test the resulting points by finding $f(x,y)$ for each solution.

Double Integrals: Up-Down, Left-Right

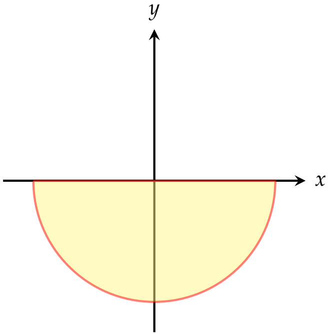
Set up but do not evaluate the following double integrals:

- $\iint_D (2yx^2 + 9y^3)\, dA$ if D is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$
- $\iint_D (10x^2y^3 - 6)\, dA$ if D is the region bounded by $x = -2y^2$ and $x = y^3$

Please see the link below for solutions

Source: Paul's Online Math Notes, examples for §15.3

Double Integrals: Polar

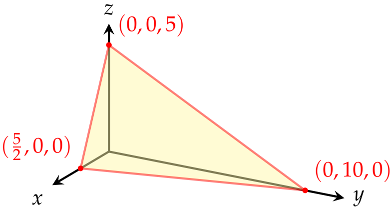


Find $\iint_D \sqrt{1+4x^2+4y^2}\,dA$ if D is the bottom half of $x^2+y^2=16$

$$\iint_D \sqrt{1+4x^2+4y^2}\,dA = \int_{\pi}^{2\pi} \sqrt{1+4r^2}\,r\,dr\,d\theta$$

Source: Paul's Online Math Notes, Practice Problems for §15.4

Triple Integrals: Cartesian



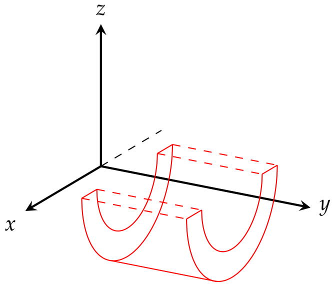
Find $\iiint_E 6z^2\,dV$ if E is the region in the first octant below the plane $4x + y + 2z = 10$.

See the reference below for solutions

Source: Paul's Online Math Notes, practice problems for §15.5

Triple Integrals: Cylindrical

Set up but do not evaluate the triple integral $\iiint_E e^{-x^2-z^2}\,dV$ if E is the region between the two cylinders $x^2+z^2=4$ and $x^2+z^2=9$ with $1 \le y \le 5$ and $z \le 0$.



The region is described by

$$\pi \le \theta \le 2\pi, \quad 2 \le r \le 3, \quad 1 \le y \le 5$$

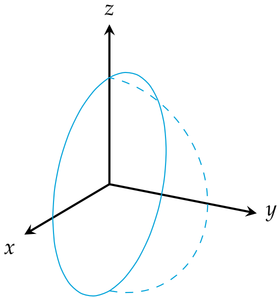
where $x = r\cos\theta, z = r\sin\theta$. The triple integral is

$$\int_{\pi}^{2\pi}\int_{2}^{3}\int_{1}^{5} e^{-r^2}\,dy\,r\,dr\,d\theta$$

Source: Paul's Online Math Notes, Practice Problems for §15.6

Triple Integrals: Spherical

Find $\iiint_E (x^2 + y^2)\, dV$ if E is the part of the sphere of radius 2 centered at $(0, 0, 0)$ with $y \geq 0$.



The region is described by

$$0 \leq \rho \leq 2, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq \pi$$

In spherical coordinates

$$x^2 + y^2 = \rho^2 \sin^2 \varphi$$

So the integral is given by

$$\int_0^{\pi} \int_{\pi}^{2\pi} \int_0^2 \rho^2 \sin^2 \varphi \; \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$$

Source: Paul's Online Math Notes, Practice Problems for §15.7