Math 213 - Semester Review, Part II

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December 6, 2023

Unit D: Vector Calculus

- November 17 - Gradient, Divergence, Curl
- November 20 - The Divergence Theorem
- November 27 - Green's Theorem
- November 29 - Stokes' Theorem, Part I
- December 1 - Stokes' Theorem, Part II
- December 4 - Final Review
- December 6 - Final Review

Final Exam Reminders

Your final exam takes place on Thursday, December 14, 6:00 PM-8:00 PM

Section 011	CP 139
Sections 012-014	CP 153

The final exam will consist of:

- 10 multiple choice questions covering Units A-D
- 4 free response questions focussing on Unit D together with material on conservative vector fields, parametrized surfaces, and surface integrals from from Unit C.

You are allowed one sheet of notes on notebook-sized paper. Notes on both sides are OK.

The Big Picture

What was this course about?

Unit A Three-dimensional space, vector algebra, equations of lines and planes, space curves, quadric surfaces

Unit B Differential calculus for functions of several variables: partial derivatives, chain rule, tangent planes and normal lines, the gradient, maxima and minima, Lagrange multiplier method

Unit C Integral calculus for functions of several variables: double integrals in Cartesian and polar coordinates, triple integrals in Cartesian, cylindrical, spherical coordinates, general coordinate changes. "Preview" topics: vector fields, line integrals, conservative vector fields. Integrals of scalar functions and vector fields over parametrized surfaces.

Unit D Vector Calculus: gradient, divergence, and curl. "Fundamental theorems" of vector calculus: the divergence theorem, Green's theorem, and Stokes' Theorem.

Scalar Line Integrals

If C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ a \leq t \leq b$ and $f(x,y,z)$ is a *scalar* function

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))|\mathbf{r}'(t)|\,dt$$

Example: Set up but don't evaluate $\int_C (x^2+y^2+z^2)\,ds$ for

$$C : x = t, \quad y = \cos 2t, \quad z = \sin 2t.$$

Vector Field Line Integrals

If C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k},\ a \le t \le b$ and $\mathbf{F}(x,y,z)$ is a vector field, then

$$\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(x(t),y(t),z(t)) \cdot \mathbf{r}'(t)\, dt$$

Example: Set up but don't evaluate $\int_C \left((x^2+y^2)\mathbf{i} + xz\mathbf{j} + (y+z)\mathbf{k}\right) \cdot d\mathbf{r}$ if

$$C : x = t^2 \quad y = t^3, \quad z = -2t$$

Time Out: Parametrized Surfaces I

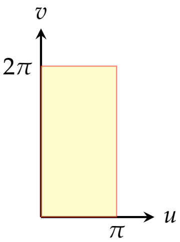
A parametrized surface is a surface S given by a function

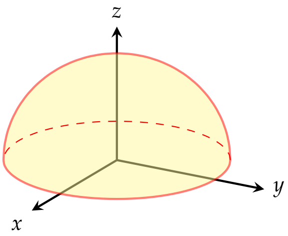
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

for (u,v) in a domain D of the uv plane.

Example: The hemisphere of radius 2 above the xy plane is given by

$$\mathbf{r}(u,v) = 2\sin(u)\cos(v)\mathbf{i} + 2\sin(u)\sin(v)\mathbf{j} + 2\cos(u),$$
$$(u,v) \in \{(u,v) : 0 \leq u \leq \pi,\ 0 \leq v \leq 2\pi\}$$





Time Out: Parameterized Surfaces II

A parametrized surface is a surface S given by a function

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

for (u,v) in a domain D of the uv plane.

If

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

then the *surface element* is

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

and the *unit normal* is

$$\widehat{\mathbf{n}} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

The *vector surface element* is

$$d\mathbf{S} = \widehat{\mathbf{n}}\, dS = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

Scalar Surface Integrals

If S is parametrized by

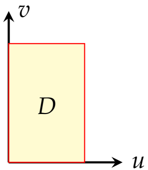
$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in R$$

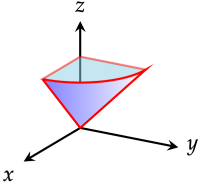
and $f(x,y,z)$ is a scalar function, then

$$\iint_S f(x,y,z)\, dS = \iint_R f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$$

Example: Find $\iint_S xyz\, dS$ if

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, u\rangle, \quad 0 \le u \le 1, \quad 0 \le v \le \pi/2$$





Vector Surface Integrals

If S is parametrized by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in R$$

and $\mathbf{F}(x,y,z)$ is a vector field, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

Example: Set up but do not evaluate the surface integral

$$\iint_S (xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}) \cdot d\mathbf{S}$$

if S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \leq x, y \leq 1$ and has upward orientation.

See next slide for graphs!

Vector Surface Integrals

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_R \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

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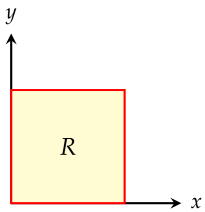
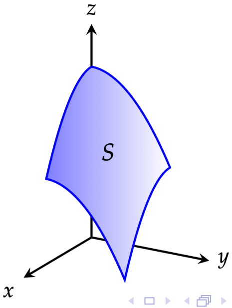




What do Line and Surface Integrals Measure?

If C is a *closed,simple* path, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is called the *circulation* of the vector field $\mathbf{F}$ around the curve C

If S is an oriented, closed surface with piecewise smooth boundary, and $\mathbf{n}$ is an *outward* normal, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

is the *flux* of the vector field out of the closed surface S

The *circulation* enters in Stokes' Theorem and the *flux* enters in the Divergence Theorem.

Gradient, Divergence, and Curl

The *gradient*

$$(\nabla f)(x,y,z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

gives the magnitude and direction of greatest change of f at (x,y,z)

If

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

is a vector field:

The *divergence*

$$(\nabla \cdot \mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the net flux of **F** out of an infinitesimal sphere per unit time per unit volume

Gradient, Divergence, and Curl

The *curl* of a vector field

$$\mathbf{v}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

is

$$(\nabla \times \mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial f}{\partial z} \\ P & Q & R \end{vmatrix}$$

If $\widehat{\mathbf{n}}$ is a unit vector, then

$$(\nabla \times \mathbf{v})(x,y,z) \cdot \widehat{\mathbf{n}}$$

measures twice the angular velocity of a small paddlewheel placed at (x,y,z) with axis of rotation in the $\widehat{\mathbf{n}}$ direction

Divergence Theorem

Suppose that V is a bounded volume enclosed by an oriented closed surface S. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{F}\, dV$$

Remember that

$$d\mathbf{S} = \widehat{\mathbf{n}}\, dS = (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv$$

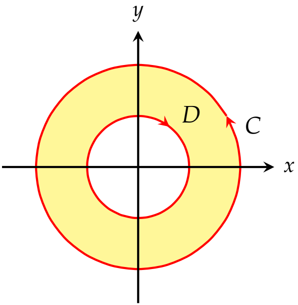
for a parametrization of S, where $\mathbf{r}_u \times \mathbf{r}_v$ points in the direction of the outward normal

Green's Theorem

If D is a domain with oriented piecewise smooth boundary C, and $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a vector field, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

where the boundary is oriented with "surface to the left"

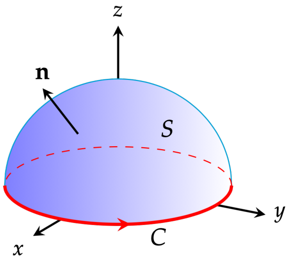


Stokes' Theorem

If S is an oriented surface with oriented boundary C, and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \widehat{\mathbf{n}}\, dS$$

where C is oriented so that a walker with the orientation of $\mathbf{n}$ walks around S with surface to the left.



It's been great to work with you. . .

Good Luck on Finals!