Math 213 - Planes

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August 30, 2023

Unit A: Vectors, Curves, and Surfaces

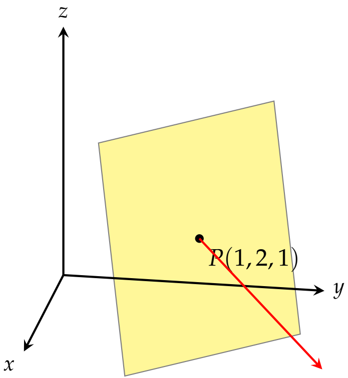
- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

Experimental Mathematics, Part I

First, we'll look at some sample equations of planes using the Geogebra 3D Calculator. You can follow along (and experiment for yourself) on your laptop or notepad.

Planes

To identify a plane uniquely you need to specify a *normal vector* and a *point on the plane*



A point on the plane is $P(1,2,1)$

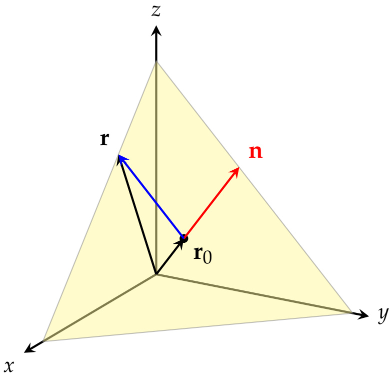
A vector normal to the plane is $\langle -2, -2, 1 \rangle$

Note: You can check that the equation of this plane is

$$-2x - 2y + z = -5$$

using the normal vector and the given point on the plane

Equation of a Plane



Suppose that a plane passes through

$$P(x_0, y_0, z_0)$$

and

$$\mathbf{n} = \langle a, b, c \rangle$$

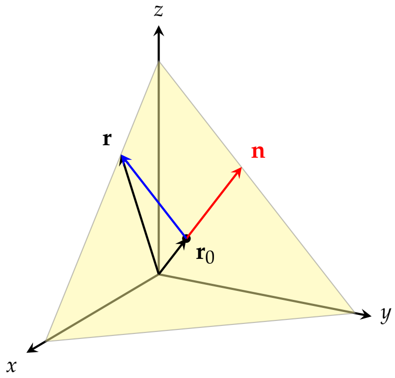
is a normal vector. If $\mathbf{r}$ is any other point on the plane

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

Here

$$\mathbf{r} - \mathbf{r}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle.$$

Equation of a Plane



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$$\mathbf{r} - \mathbf{r}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle.$$

If we expand the dot product we get

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Equation of a Plane

For a plane with normal $\mathbf{n} = \langle a, b, c, \rangle$ passing through the point (x_0, y_0, z_0):

$$\begin{aligned} \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) &= 0 && \text{Vector Equation} \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 && \text{Scalar Equation} \\ ax + by + cz &= d && \text{Easy Equation} \end{aligned}$$

Suppose a plane has equation

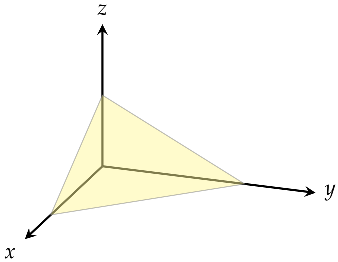
$$2x + 2y + 4z = 4$$

What is a normal vector to the plane?

What points does it pass through?

Equation of Plane

$$2x + 2y + 4z = 4$$

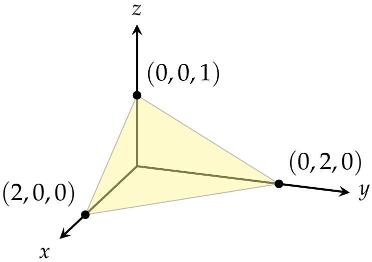


What is a normal vector?
$\langle 2, 2, 4 \rangle$

What points does it pass through?
You can find the plane's intersection with the x, y, and z axes, for example.

Equation of Plane

$$2x + 2y + 4z = 4$$



What is a normal vector?
$\langle 2, 2, 4 \rangle$

What points does it pass through?
You can find the plane's intersection with the x, y, and z axes, for example.

- You can read off the normal vector from the equation!
- You can find the points where the plane intersects the x, y and z axes

Puzzler #1

Easy equation of a plane:

$$ax + by + cz = d$$

where $\mathbf{n} = \langle a, b, c \rangle$ is a normal vector.

Find the equation of a plane with normal vector $\mathbf{n} =< 2, 1, 2 >$ passing through the point $(1, -2, 4)$.

From the equation in the inset, we have

$$2x + y + 2z = d$$

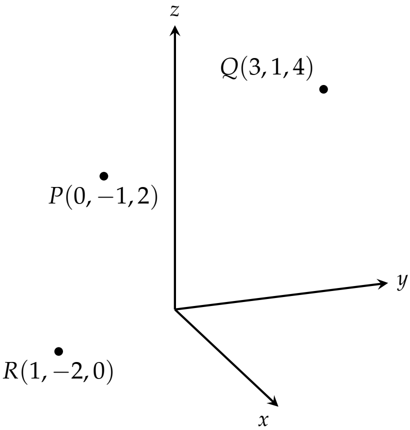
and we can find d by substituting in the point $(1, -2, 4)$

$$2(1) + (-2) + 2(4) = 8.$$

So, the equation of the plane is

$$2x + y + 2z = 8.$$

Puzzler #2



Find the equation of the plane through the points $(1,-2,0)$, $(3,1,4)$, and $(0,-1,2)$

(From Paul's Online Math Notes)

Two vectors in the plane are

$$\overrightarrow{PQ} = \langle 3,2,2 \rangle$$
$$\overrightarrow{PR} = \langle 1,-1,-2 \rangle$$

so a normal vector is

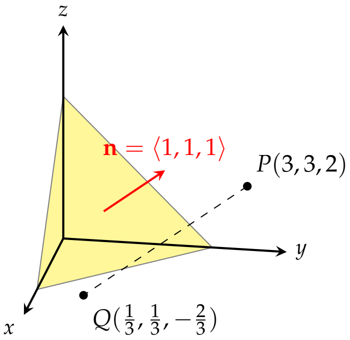
$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -2,8,-5 \rangle$$

Hence

$$-2x + 8y - 5z = d$$

and using $P(0,-1,2)$ in this equation we find $d = -18$.

Distance from a Point to a Plane



Find the distance from the point $P(3,3,2)$ to the plane

$$x + y + z = 2$$

We'll follow a line normal to the plane from $(3,3,2)$ to the plane: the equation of the line is

$$\langle x(t), y(t), z(t)\rangle = \langle 3+t, 3+t, 2+t\rangle$$

and hits the plane when

$$8 + 3t = 2$$

or $t = -2$ (see the point Q). The normal has length $\sqrt{3}$ so the distance travelled from P to Q is $2\sqrt{3}$.

Experimental Mathematics, Part II

Let's use the Geogebra 3D Calculator again to plot two planes at the same time.

Suggestion 1: Plot

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 4 \end{aligned}$$

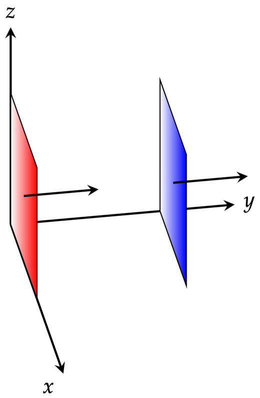
Do these planes intersect?

Suggestion 2: Plot

$$\begin{aligned} 2x - y + z &= 4 \\ x + 2y + z &= 2 \end{aligned}$$

Do these planes intersect?

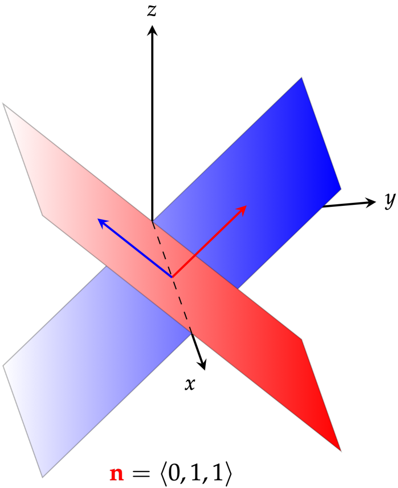
Parallel and Intersecting Planes



Two planes are either:

- *Parallel*, if their normal vectors are parallel, or

Parallel and Intersecting Planes



$\color{red}{\mathbf{n}} = \langle 0, 1, 1 \rangle$
$\color{blue}{\mathbf{n}} = \langle 0, -1, 1 \rangle$

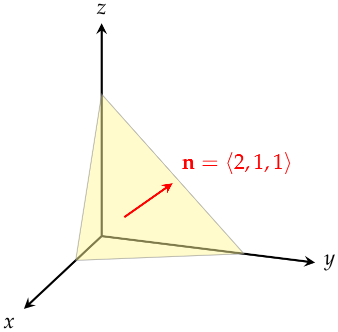
Two planes are either:

- *Parallel*, if their normal vectors are parallel, or
- *Intersecting* along a line perpendicular to their normal vectors

At what angle do the two planes shown intersect?

The dot product of the two normals is zero so the planes intersect at an angle of $\pi/2$ or 90°

Puzzler

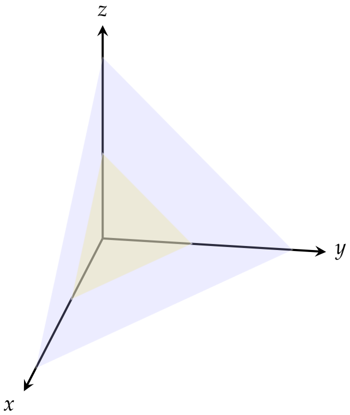


Find all planes parallel to the plane

$$2x + y + z = 4$$

and at distance 2 from the original plane.

Distance Between Planes



Find the distance between the planes

$$x + 2y + 2z = 4$$

and

$$2x + 4y + 4z = 17$$

Reminders for the Week of August 28-September 1

- WebWork A2 due on Wednesday 8/30 by 11:59 PM
- Recitation on planes Thursday 8/31
- Quiz # 1 on coordinate systems and vectors due on Thursday 8/31 at 11:59 PM
- Read CLP 3, section 1.5 for Friday 9/1