Math 213 - Lines in Space

Peter Perry

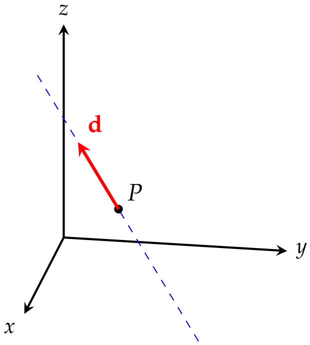
September 1, 2023

Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

Introducing Bill the Heron

Equations of Lines



A line is space is determined by:

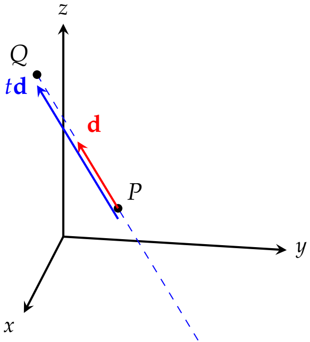
- A point P on the line
- A vector $\mathbf{d}$, the *displacement vector*, that points along the line

In the line shown,

$$P = P(3/2, 1, 1)$$
$$\mathbf{d} = (1/4, -1, 1)$$

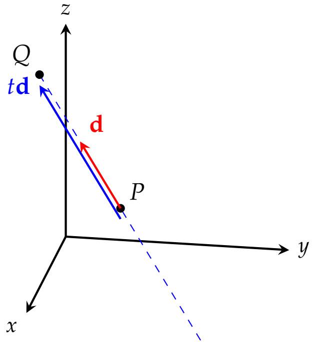
How can we write an equation for this line?

Equations of Lines



You can get from P to any other point Q on the line by a vector displacement $t\mathbf{d}$ for some real number t

Equations of Lines



You can get from P to any other point Q on the line by a vector displacement $t\mathbf{d}$ for some real number t

If

$$P = P(3/2, 1, 1)$$

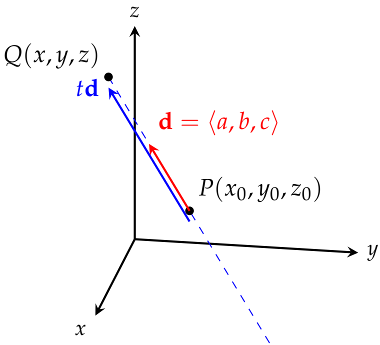
and

$$\mathbf{d} = \langle 1/4, -1, 1 \rangle,$$

the coordinates of Q are given by

$$x = 3/2 + (1/4)t$$
$$y = 1 + (-1)t$$
$$z = 1 + (1)t$$

The Parametric Equations of a Line



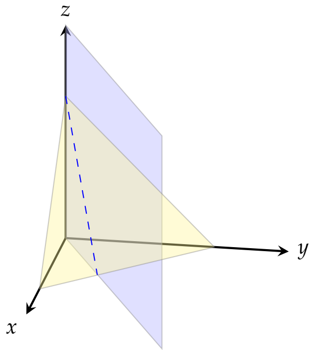
The parametric equations of a line are:

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle$$

or

$$\begin{aligned} x - x_0 &= ta \\ y - y_0 &= tb \\ z - z_0 &= tc \end{aligned}$$

Puzzler #1



The planes

$$x + y + z = 2$$
$$x - 2y = 0$$

are shown at left. Find the equation of their line of intersection.

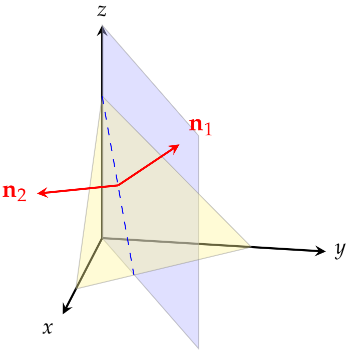
We can use z as the parameter and solve for x and y:

$$x + y - 2 = -z \quad \Rightarrow$$
$$x = 2y$$

$$x = -\frac{2}{3}z + \frac{4}{3}$$
$$y = -\frac{1}{3}z + \frac{2}{3}$$

so $\langle x - \frac{4}{3}, y - \frac{2}{3}, z\rangle = z\langle -\frac{2}{3}, -\frac{1}{3}, 1\rangle$

Puzzler #2



The planes

$$x + y + z = 2$$
$$x - 2y = 0$$

have normal vectors

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle,$$
$$\mathbf{n}_2 = \langle 1, -2, 0 \rangle.$$

Find another equation for the line of intersection.

A displacement vector for the line is

$$\mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$$

and the line passes through the point $(1/2, 3/2, 0)$ (use the equations above with $z = 0$)

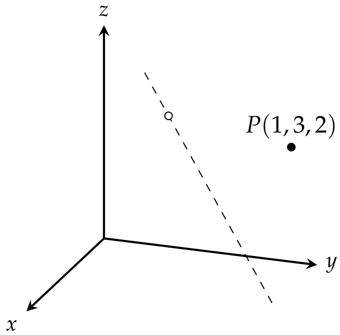
So we get

$$\langle x - \frac{1}{2}, y - \frac{3}{2}, z \rangle = t\langle 2, 1, -3 \rangle$$

or

$$x - \frac{1}{2} = 2t, \; y - \frac{3}{2} = t, \; z = -3t.$$

Puzzler #3



Find the distance from the point

$$P = (1,3,2)$$

to the line

$$L : \langle x-1, y-2, z-1\rangle = t\langle 1,0,1\rangle$$

Another form of the equation for the line is

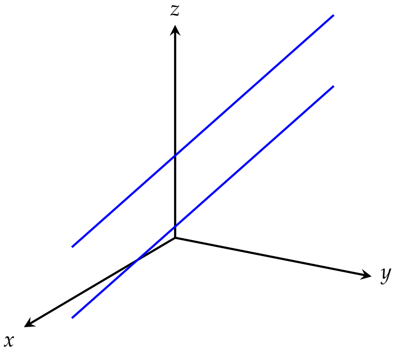
$$x(t) = 1+t, \quad y(t) = 2, \quad z(t) = 1+t.$$

The squared distance from P to any point on the line is

$$\begin{aligned} f(t) &= (1-(1+t))^2 + (3-2)^2 + (2-(1+t))^2 \\ &= t^2 + 1 + (1-t)^2 \\ &= 2t^2 - 2t + 2 \end{aligned}$$

Since $f'(t) = 4t - 2$, the minimum value occurs at $t = 2$ where $f(2) = 6$. Hence the distance from P to the line is $\sqrt{6}$.

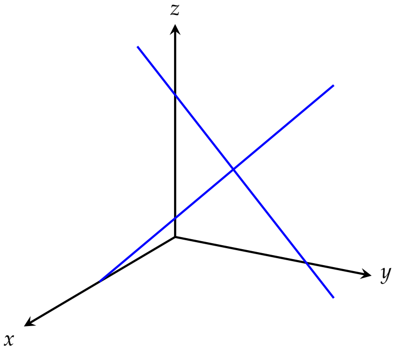
What Can Two Lines Do?



In three dimensions, lines can be:

- Parallel

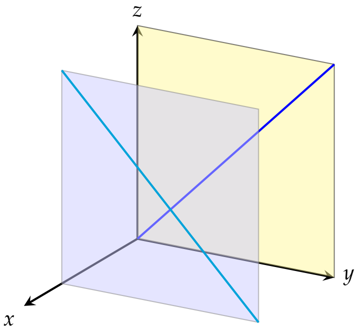
What Can Two Lines Do?



In three dimensions, lines can be:

- Parallel
- Intersecting, or

What Can Two Lines Do?

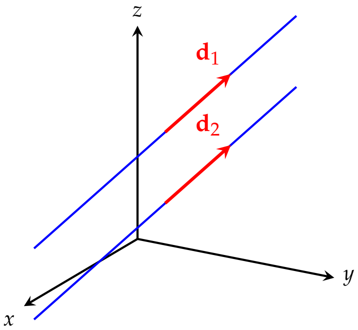


In three dimensions, lines can be:

- Parallel
- Intersecting, or
- Skew

What Can Two Lines Do?

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle = t\mathbf{d}$$

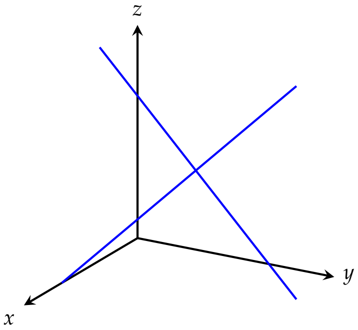


How do you tell when two lines are:

- parallel - parallel vectors

What Can Two Lines Do?

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle = t\mathbf{d}$$

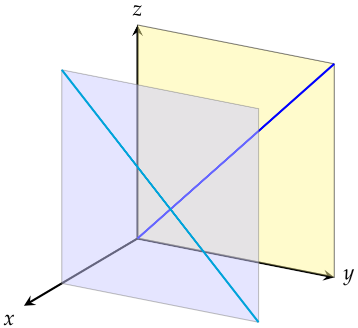


How do you tell when two lines are:

- parallel - parallel vectors
- intersecting - point in common

What Can Two Lines Do?

$$\langle x - x_0, y - y_0, z - z_0 \rangle = t\langle a, b, c \rangle = t\mathbf{d}$$



How do you tell when two lines are:

- parallel - parallel vectors
- intersecting - point in common
- skew - neither parallel nor intersecting

Finding out What Two Lines Do

Two lines are:

- parallel if the **d** vectors are parallel
- intersecting if they have a point in common
- skew if none of the above

Are the two lines

$$\begin{aligned} x-1 &= 2s \\ y-2 &= -s \\ z-3 &= 3s \end{aligned} \qquad \begin{aligned} x-3 &= 4t \\ y-4 &= -2t \\ z-5 &= 6t \end{aligned}$$

parallel, intersecting, or skew?

The displacement vectors for the two lines are

$$\mathbf{d}_1 = \langle 2, -1, 3 \rangle \text{ and } \mathbf{d}_2 = \langle 4, -2, 6 \rangle.$$

Since these vectors are parallel ($\mathbf{d}_2 = 2\mathbf{d}_1$), the lines are parallel.

Finding out What Two Lines Do

Two lines are:

- parallel if the **d** vectors are parallel
- intersecting if they have a point in common
- skew if none of the above

Are the two lines

$$\begin{aligned} x-1&=0 \\ y&=s \\ z&=-s \end{aligned} \qquad \begin{aligned} x-1&=t \\ y-3&=-3t \\ z+3&=3t \end{aligned}$$

parallel, intersecting, or skew?

The displacement vectors are $\langle 0,1,-1\rangle$ and $\langle 1,-3,3\rangle$ so the vectors aren't parallel. Solve the equations

$$0=t, \quad s=3-3t, \quad -s=3t-3$$

to find $t=0$, $s=3$ and a point of intersection at $\langle 1,3,-3\rangle$.

Finding out What Two Lines Do

Two lines are:

- parallel if the **d** vectors are parallel
- intersecting if they have a point in common
- skew if none of the above

Are the lines

$$x - 2 = s \qquad x - 2 = -t$$
$$y - 2 = 0 \qquad y + 2 = 0$$
$$z - 3 = 2s \qquad z - 2 = 4t$$

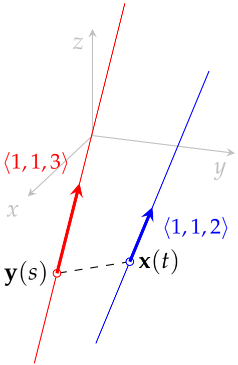
parallel, intersecting, or skew?

These lines are not parallel (check the displacement vectors!). If we check for an intersection we get the equations

$$s = -t, \quad 2 = -2, \quad 2s + 3 = 2 + 4t$$

which are not consistent (the second one especially!). These lines are skew.

The Distance Between Lines



Find the distance between the following two lines:

$$\mathbf{x}(t) = \langle t, 2+t, 2t-1 \rangle$$

and

$$\mathbf{y}(s) = \langle 1+s, s, -1+3s \rangle$$

Hint: At the closest approach, $\mathbf{x}(t) - \mathbf{y}(s)$ should be perpendicular to the direction vectors of *both* lines

See the next page for the solution

Problem courtesy of this YouTube video

The Distance Between Two Lines

From $\mathbf{x}(t) = \langle t, 2+t, 2t-1 \rangle, \quad \mathbf{y}(s) = \langle 1+s, s, -1+3s \rangle$ we get

$$\mathbf{x}(t) - \mathbf{y}(s) = \langle t-s-1, 2+t-s, 2t-3s \rangle.$$

The two displacement vectors are $\mathbf{d}_1 = \langle 1,1,2 \rangle$ and $\mathbf{d}_2 = \langle 1,1,3 \rangle$. From the two conditions $\mathbf{d}_1 \cdot (\mathbf{x}(t) - \mathbf{y}(s)) = 0$, $\mathbf{d}_2 \cdot (\mathbf{x}(t) - \mathbf{y}(s)) = 0$, we get

$$(t-s-1) + (2+t-s) + (4t-6s) = 0$$
$$(t-s-1) + (2+t-s) + (6t-9s) = 0$$

or, simplifying

$$6t - 8s = -1$$
$$8t - 11s = -1$$

You can solve these equations to find $s = -1$, $t = -3/2$ so that

$$\mathbf{x}(-3/2) - \mathbf{y}(-1) = \langle -3/2, 1/2, -4 \rangle - \langle 0, -1, -4 \rangle = \langle -3/2, -1/2, 0 \rangle$$

The distance between the two lines is $\sqrt{(-3/2)^2 + (-1/2)^2} = \sqrt{10}/2$

Reminders for the Week of September 5-8

- Happy Labor Day, Monday 9/4!
- Read CLP 3 section 1.6 for Wednesday 9/6
- Homework A3 on Equations of Lines and Planes due Wednesday 9/6 at 11:59 PM
- Quiz #2 on Equations of LInes and Planes due Thursday 9/7 at 11:59 PM
- Read CLP 4, sections 1.2 and 1.6 for Friday 9/8
- Homework A4 on Curves and Tangent Vectors due Friday 9/8