# Math 213 - Plane and Space Curves

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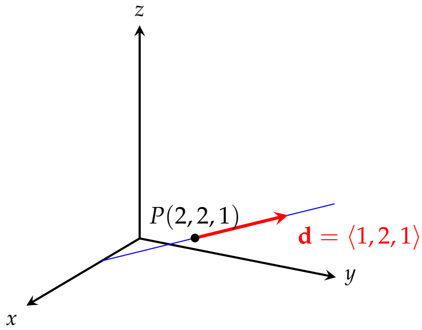
September 6, 2023

# Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

# Parameterized Curves




The parametric equations

$$x(t) = 2 + t$$
$$y(t) = 2 + 2t$$
$$z(t) = 1 + t$$

where

$$0 \leq t \leq 3$$

define a line in three-dimensional space.

We're going to study other *parameteric curves* where we're given a *vector function*

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

and an allowed range of $t$-values

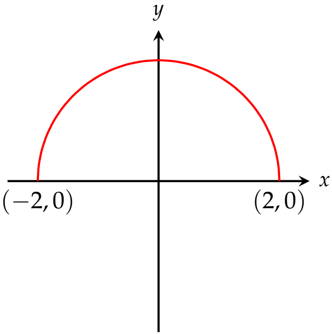
$$a \leq t \leq b$$

.

# Parametric Curves - $xy$ Plane

Parametric Curve in the $xy$ Plane:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \qquad a \le t \le b$$



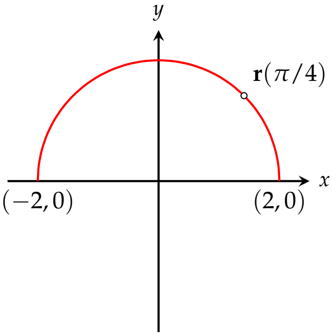

The curve at left is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle, \quad 0 \le t \le \pi.$$

# Parametric Curves - $xy$ Plane

Parametric Curve in the $xy$ Plane:

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \qquad a \leq t \leq b$$




The curve at left is

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle, \quad 0 \leq t \leq \pi.$$

$$\mathbf{r}(\pi/4) = \langle \sqrt{2}, \sqrt{2} \rangle$$

# Parametric Curves - $xy$ Plane

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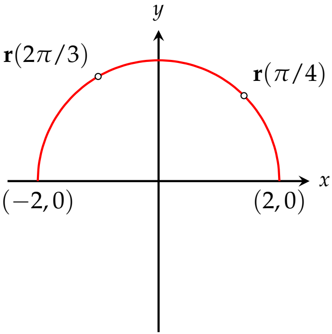


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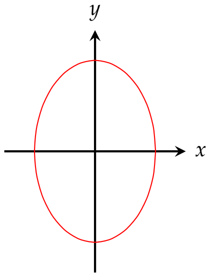
$$\mathbf{r}(\pi/4) = \langle \sqrt{2}, \sqrt{2} \rangle$$

$$\mathbf{r}(2\pi/3) = \langle -1, \sqrt{3} \rangle$$

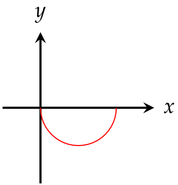
# Parametric Curves - Mix and Match

Can you match these parametric curves with their graphs?

$$\mathbf{r}(t) = \langle 1 + \cos(t), \sin(t) \rangle, \qquad \pi \le t \le 2\pi$$
$$\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle, \qquad 0 \le t \le 2\pi$$
$$\mathbf{r}(t) = \langle t, t^2 \rangle, \qquad 0 \le t \le 2$$




$\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle$

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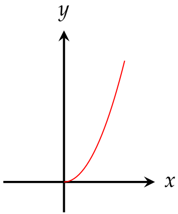
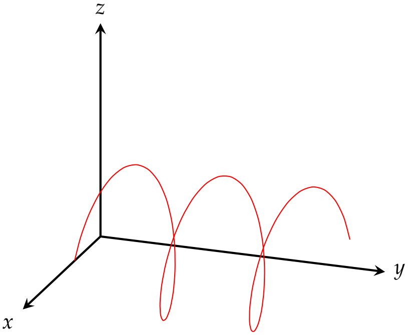



$\mathbf{r}(t) = \langle t, t^2 \rangle$

# Parametric Curves in Space



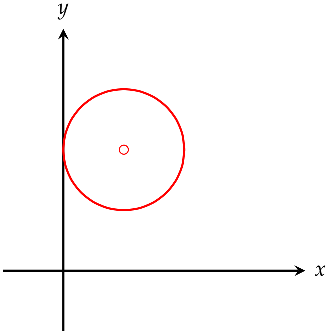

Which of these is the correct formula for $\mathbf{r}(t)$?

$$\mathbf{r}(t) = \langle t, \cos(5t), \sin(5t) \rangle$$
$$\mathbf{r}(t) = \langle \cos(5t), t, \sin(5t) \rangle$$
$$\mathbf{r}(t) = \langle t, \sin(5t), \cos(5t) \rangle$$
$$\mathbf{r}(t) = \langle \sin(5t), t, \cos(5t) \rangle$$

Here $0 \leq t \leq \pi$.

$\mathbf{r}(t) = \langle \cos(5t), t, \sin(5t) \rangle$

# Unparametrization




Find an equation in $x$ and $y$ for the curve

$$x(t) = 1 + \cos(2t)$$
$$y(t) = 2 - \sin(2t)$$

Since

$x(t) - 1 = \cos(2t), \quad y(t) - 2 = -\sin(2t)$

we get

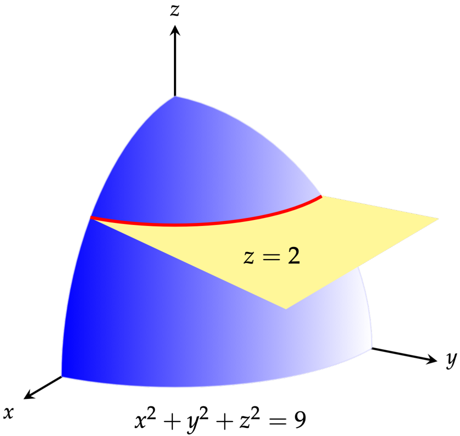
$$(x(t) - 1)^2 + (y(t) - 2)^2 = 1$$

or

$$(x - 1)^2 + (y - 2)^2 = 1$$

which is the equation of a circle of radius 1 with center at $(1, 2)$.

# Intersection of Surfaces




Find a parametric equation for the curve where the surfaces

$$x^2 + y^2 + z^2 = 9$$

and

$$z = 2$$

intersect.

The intersection given by the equation $x^2 + y^2 + 4 = 9$ or

$$x^2 + y^2 = 5,$$

the equation a circle of radius $\sqrt{5}$. We can parametrize the circle by

$$x(t) = \sqrt{5}\cos t, \quad y(t) = \sqrt{5}\sin t, \quad z(t) = 2.$$

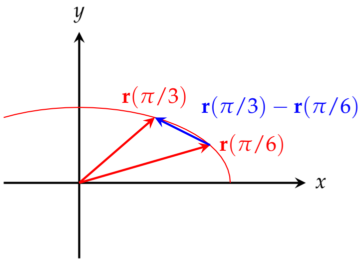
# Derivatives

The derivative of a function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ is

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Practically speaking,

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$$




The difference

$$\mathbf{r}(t+h) - \mathbf{r}(t)$$

gives the motion along the curve between $t$ and $t+h$

# Derivatives

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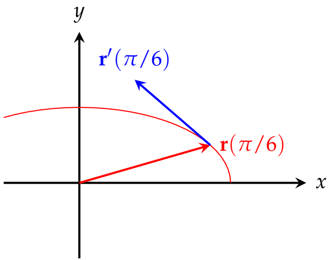



The derivative $\mathbf{r}'(t)$ gives the *velocity* along the curve

$\mathbf{r}'(t)$ is also called a *tangent vector*

# Derivative Practice

Find the following derivatives if:

$$\gamma(t) = t^2$$
$$\mathbf{v}(t) = \mathbf{i} + e^t\mathbf{j} + e^{2t}\mathbf{k}$$
$$\mathbf{w}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$$

$$\frac{d}{dt}\left(\gamma(t)\mathbf{w}(t)\right) = (2t)\mathbf{i} + (t^2 + 2t)e^t\mathbf{j} + (2t^2 + 2t)e^{2t}\mathbf{j}$$
$$\frac{d}{dt}\left(\mathbf{v}(t) \cdot \mathbf{w}(t)\right) = 1 + (t^2 + 2t)e^t + 2e^{2t}$$
$$\frac{d}{dt}\left(\mathbf{v}(t) \times \mathbf{w}(t)\right) = (e^t - (2t^2 + 2t)e^{2t})\mathbf{i} + (2 - t)e^{2t}\mathbf{j} + (2t - (t+1)e^t)\mathbf{k}$$

# Derivative Rules

If $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are differentiable vector functions, $\alpha$ and $\beta$ are scalars, and $\gamma(t)$ and $s(t)$ are differentiable functions:

$$\frac{d}{dt}\left(\alpha\mathbf{a}(t) + \beta\mathbf{b}(t)\right) = \alpha\mathbf{a}'(t) + \beta\mathbf{b}'(t) \qquad \text{linear combination}$$

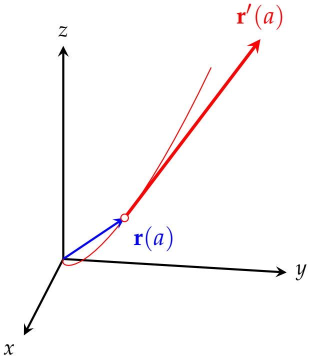
$$\frac{d}{dt}\left(\gamma(t)\mathbf{a}(t)\right) = \gamma'(t)\mathbf{a}(t) + \gamma(t)\mathbf{a}'(t) \qquad \text{product rule}$$

$$\frac{d}{dt}\left(\mathbf{a}(t)\cdot\mathbf{b}(t)\right) = \mathbf{a}'(t)\cdot\mathbf{b}(t) + \mathbf{a}(t)\cdot\mathbf{b}'(t) \qquad \text{product rule } (\cdot)$$

$$\frac{d}{dt}\left(\mathbf{a}(t)\times\mathbf{b}(t)\right) = \mathbf{a}'(t)\times\mathbf{b}(t) + \mathbf{a}(t)\times\mathbf{b}'(t) \qquad \text{product rule } (\times)$$

$$\frac{d}{dt}\left(\mathbf{a}(s(t))\right) = s'(t)\mathbf{a}'(s(t)) \qquad \text{(chain rule)}$$

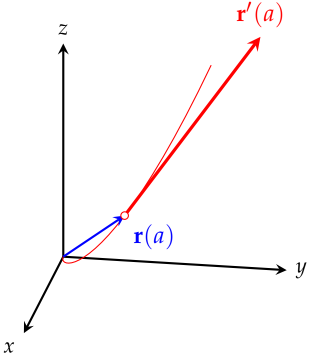
# Tangent, Unit Tangent, Arc Length




$\mathbf{r}'(a)$ is the *tangent vector* to $\mathbf{r}(t)$ at $t = a$

$|\mathbf{r}'(a)|$ is the *instantaneous speed* along the curve $\mathbf{r}(t)$ at time $t = a$

# Tangent, Unit Tangent, Arc Length

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$|\mathbf{r}'(a)|$ is the *instantaneous speed* along the curve $\mathbf{r}(t)$ at time $t = a$

How can we find a unit vector tangent to $\mathbf{r}(t)$ at $t = a$?

Find the unit tangent to

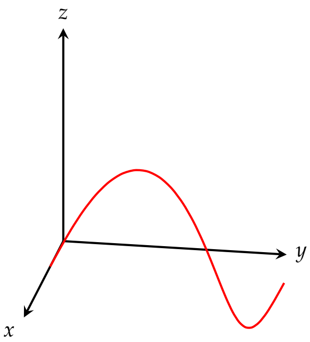
$$\mathbf{r}(t) = \langle t, t^2, t^2 \rangle$$

at $t = 1$

$\mathbf{r}'(1) = \langle 1, 2, 2 \rangle$ so the unit tangent is

$$\mathbf{T}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

# Tangent, Unit Tangent, Arc Length




$\mathbf{r}'(a)$ is the *tangent vector* to $\mathbf{r}(t)$ at $t = a$

$|\mathbf{r}'(a)|$ is the *instantaneous speed* along the curve $\mathbf{r}(t)$ at time $t = a$

How can we find the distance travelled along the curve $\mathbf{r}(t)$ between $t = a$ and $t = b$?

Find the distance travelled from $t = 0$ to $t = \pi$ along

$$\mathbf{r}(t) = \langle \cos(2t), t, \sin(2t) \rangle$$

Since $\mathbf{r}'(t) = \langle -2\sin(2t), 1, 2\cos(2t) \rangle$, $|\mathbf{r}'(t)| = \sqrt{5}$ (constant speed), and the distance travelled is $\pi\sqrt{5}$

# Summary

Given a parameterized curve $\mathbf{r}(t)$ for $a \leq t \leq b$:

- The tangent vector to the curve at time $t = c$ is given by $\mathbf{r}'(c)$
- The unit tangent vector to the curve at time $t = c$ is given by

$$\mathbf{T}(c) = \frac{\mathbf{r}'(c)}{|\mathbf{r}'(c)|}$$

- If $s(t)$ denotes the distance travelled along the curve between times $a$ and $t$, then

$$s'(t) = |\mathbf{r}'(t)|$$

- The distance travelled from $t = a$ to $t = b$ is given by

$$s = \int_a^b |\mathbf{r}'(t)|\, dt$$

# Velocity and Acceleration

If $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ is the position of a particle at time $t$, then:

- $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity of the particle at time $t$
- $|\mathbf{r}'(t)|$ is the speed of the particle at time $t$
- $\mathbf{a}(t) = \mathbf{v}'(t)$ is the acceleration of the particle at time $t$

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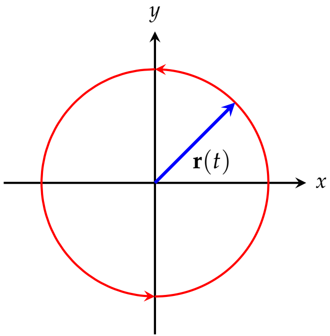


A particle moves along the curve

$$\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t) \rangle.$$

Find the velocity and acceleration.

$$\mathbf{v}(t) = \langle -2\sin(t), 2\cos(t) \rangle$$
$$\mathbf{a}(t) = \langle -2\cos(t), -2\sin(t) \rangle$$

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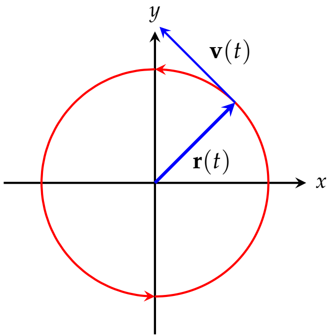


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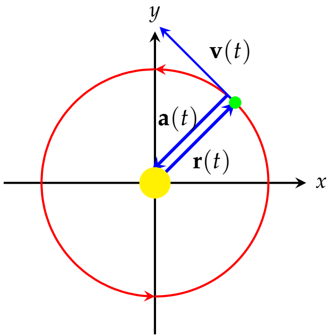


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Find the velocity and acceleration.

$$\mathbf{v}(t) = \langle -2\sin(t), 2\cos(t) \rangle$$
$$\mathbf{a}(t) = \langle -2\cos(t), -2\sin(t) \rangle$$

# Reminders for the Week of September 4-6

- Homework A3 on Equations of Lines and Planes due Wednesday 9/6 at 11:59 PM
- Quiz #2 on Equations of Lines and Planes due Thursday 9/7 at 11:59 PM
- Read CLP 4, sections 1.2 and 1.6 for Friday 9/8
- Homework A4 on Curves and Tangent Vectors due Friday 9/8