Math 213 - Integrating Along Curves

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September 8, 2023

Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

Arc Length Reminders

If C is a parameterized curve $(x(t), y(t)$ in the plane with $a \leq t \leq b$ then the arc length of C is

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_C ds.$$

The arc length function $s(t)$ is

$$s(t) = \int_a^t \sqrt{x'(t') + y'(t')^2}\, dt'.$$

If C is a parametrized curve $(x(t), y(t), z(t))$ in space with $a \leq t \leq b$ then the arc length of C is

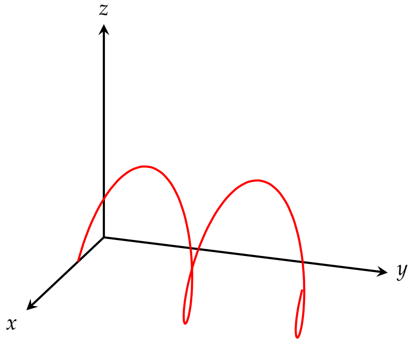
$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_C ds$$

The arc length function $s(t)$ is

$$s(t) = \int_a^t \sqrt{x'(t') + y'(t')^2 + z'(t)^2}\, dt'.$$

Arc Length

$$s(t) = \int_a^t \sqrt{x'(t')^2 + y'(t')^2 + z'(t')^2}\, dt'$$

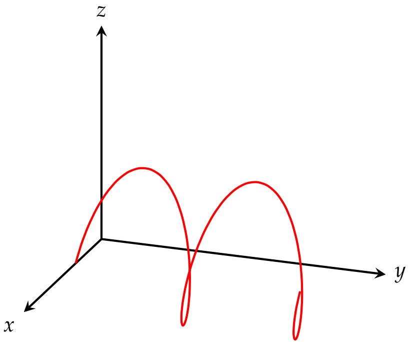


Find the arc length function for the curve

$$(x(t), y(t), z(t)) = (\cos t, t, \sin t)$$

where $0 \le t \le 4\pi$.

Parameterization by Arc Length



Recall that, for the curve

$$(x(t), y(t), z(t)) = (\cos t, t, \sin t)$$

$$0 \le t \le 4\pi,$$

the arc length function is

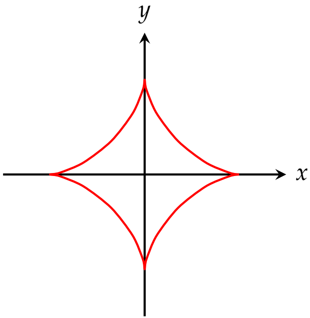
$$s(t) = \sqrt{2}t.$$

Give an equation for this curve with arc length as the parameter.

We have $T(s) = s/\sqrt{2}$ so

$$(x(s), y(s)) = (\cos(s/\sqrt{2}), s/\sqrt{2}, \sin(s/\sqrt{2}))$$

The Asteroid



At left is the curve

$$(x(t), y(t)) = (a\cos^3 t, a\sin^3 t)$$

or

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

Find the arc length function $s(t)$ for $0 \leq t \leq \pi/2$ and parametrize this curve by arc length.

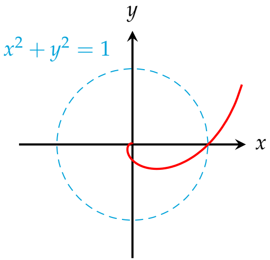
$$\sqrt{x'(t)^2 + y'(t))^2} = 3a\sqrt{\cos^2 t \sin^2 t(\sin^2 t + \cos^2 t)}$$
$$= \frac{3a}{2}\sin(2t)$$

so

$$s(t) = \frac{3a}{2}\int_0^t \sin(2t')\,dt' = \frac{3a}{4}(1 - \cos(2t))$$

Solve to t in terms of s to get $T(s) = \frac{1}{2}\arccos(3a/4 - s)$ and substitute!

The Logarithmic Spiral



At left is the curve

$$(x(t), y(t)) = (e^t \cos t, e^t \sin t), \quad -\infty < t < \infty$$

First, find the arc length of the part of the curve inside the unit circle.

Second, find the arc length function for $-\infty < t < \infty$ and reparameterize the curve by arc length.

Since $\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{2}e^t$ (check this!) we get

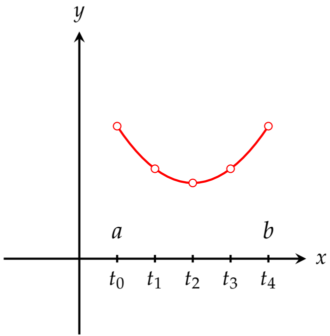
$$s(t) = \int_{-\infty}^{t} \sqrt{2}e^{t'}\, dt' = \sqrt{2}e^t$$

The curve inside the unit circle goes from $t = -\infty$ to $t = 0$, so the arc length is

$$s(0) = \sqrt{2}.$$

Second, solving for t in terms of s, we get $T(s) = \ln(s)/\sqrt{2}$. Now substitute!

The Mass of A Wire



A wire traces out a curve C

$$(x(t), y(t)), \quad a \leq t \leq b$$

and has mass $\rho(t)$ per unit length. Find the mass of the wire.

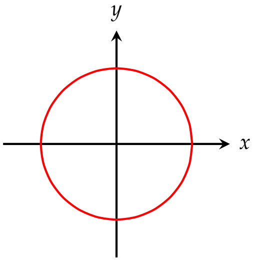
Approximation: if $\mathbf{r}(t) = \langle x(t), y(t) \rangle$,

$$\begin{aligned} M &\simeq \sum_{i=1}^{n} \rho(t_i) |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| \\ &= \sum_{i=1}^{n} \rho(t_i) \left| \frac{\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})}{t_i - t_{i-1}} \right| \Delta t \end{aligned}$$

This looks like a Riemann sum, but for what integral?

The Mass of a Wire

$$M = \int_a^b \rho(t)|\mathbf{r}'(t)|\,dt = \int_C \rho\,ds$$



A hoop traces out the curve $x^2 + y^2 = 1$ where x and y are in meters. The hoop has a mass of x^2 kg/m. What is the total mass of the hoop?

Parametrize the wire by $(x(t), y(t)) = (\cos t, \sin t)$. In this case, $|\mathbf{r}'(t)|^2 = x'(t)^2 + y'(t)^2 = 1$ and $\rho(t) = \cos^2 t$ so

$$\begin{aligned} M &= \int_0^{2\pi} \cos^2 t\,dt \\ &= \int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt \\ &= \pi \text{ kg} \end{aligned}$$

Integrals Along Curves

If $f(x, y)$ is a function of two variables, and C is a curve $(x(t), y(t))$ for $a \leq t \leq b$, then the integral of f along C is

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

What if $f(x, y, z)$ is a function of three variables and C is a curve $(x(t), y(t), z(t))$ for $a \leq t \leq b$?

Integrals Along Curves

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$$

Suppose C is the curve from $(0,1,2)$ to $(1,2,3)$. Parametrize this curve and find $\int_C xyz\,ds$.

Corrected! The parametrization is

$$(x(t),y(t),z(t)) = (t, 1+t, 2+t), \quad 0 \leq t \leq 1$$

so $x'(t)^2+y'(t)^2+z'(t)^2 = 3$. We get

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_0^1 t(1+t)(2+3t)\sqrt{3}\,dt \\ &= \sqrt{3}\int_0^1 (t^3+3t^2+2t)\,dt \\ &= \sqrt{3}\cdot\frac{9}{4}\end{aligned}$$

Reminders for the Week of September 6-8 and 11-15

- Continue reading CLP 4, sections 1.2 and 1.6
- Homework A4 due today at 11:59 PM
- Begin reading CLP 3, sections 1.7-1.9 for Wednesday and Friday of next week
- Homework A5 due Wednesday September 13 at 11:59 PM