Math 213 - Integration on Curves

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September 11, 2023

Important Reminders

- If you need disability accommodations for Exam 1, I need your accommodation letter *no later than 5 PM on Wednesday, September 13*
- If you want to request an alternate exam, you need to fill out the form here *no later than 5 PM on Friday, September 15*

Unit A: Vectors, Curves, and Surfaces

- August 21 - Points
- August 23 - Vectors
- August 25 - Dot Product
- August 28 - Cross Product
- August 30 - Equations of Planes
- September 1 - Equations of Lines
- September 6 - Curves
- September 8 - Integrating Along Curves
- September 11 - Integrating Along Curves
- September 13 - Sketching Surfaces
- September 15 - Cylinders and Quadric Surfaces

Working Backwards

$$\mathbf{v}(t) = \mathbf{r}'(t)$$
$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$$

Suppose that the acceleration $\mathbf{a}(t)$ of a moving particle is

$$\mathbf{a}(t) = \langle -16\cos(-4t), 16\sin(-4t), -t\rangle$$

and that the initial velocity and position of the particle are

$$\mathbf{v}(0) = \langle 1, 0, 1\rangle$$

and

$$\mathbf{r}(0) = \langle 1, 1, 1\rangle.$$

Find $\mathbf{v}(t)$ and $\mathbf{r}(t)$ for all t.

Solutions to Working Backwards

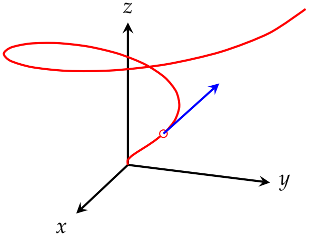
First,

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{v}(0) + \int_0^1 \mathbf{a}(s)\,ds \\ &= \langle 1,0,1\rangle + \int_0^1 \langle -16\cos(-4s), 16\sin(-4s), -s\rangle\,ds \\ &= \left\langle 1+4\sin(-4t), 4\cos(-4t)-4, \frac{1}{2}-\frac{t^2}{2}\right\rangle\end{aligned}$$

Second,

$$\begin{aligned}\mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t \mathbf{v}(s)\,ds \\ &= \langle 1,1,1\rangle + \int_0^t \langle 1+4\sin(-4s), 4\cos(-4s)-4, \frac{1}{2}-\frac{s^2}{2}\rangle\,ds \\ &= \langle 1,1,1\rangle + \langle t+\cos(-4t)-1, -\sin(-4t)-4t, \frac{t}{2}-\frac{t^3}{6}\rangle\end{aligned}$$

Working Forwards



Suppose that

$\mathbf{r}(t) = \langle t\cos(\pi t/2), t\sin(\pi t/2), t\rangle$

- Show that the path lies along the cone $x^2 + y^2 = z^2$
- Find the velocity vector at time t
- If at time $t = 1$ the particle flies off on a line L the in a direction tangent to the path, find the equation of the line.

First, $x(t)^2 + y(t)^2 = t^2\cos^2(\pi t/2) + t^2\sin^2(\pi t/2) = t^2 = z(t)^2$

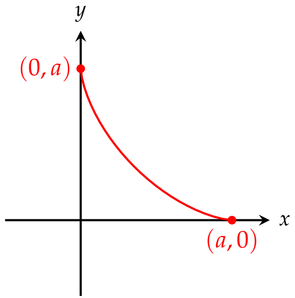
Next,

$$\mathbf{r}'(t) = \langle \cos(\pi t/2) - (\pi t/2)\sin(\pi t/2), \sin(\pi t/2) + (\pi t/2)\cos(\pi t/2), 1\rangle$$

so that $\mathbf{r}(1) = \langle 0, 1, 1\rangle$, $\mathbf{r}'(1) = \langle -\pi/2, 1, 1\rangle$ and the line L has equation

$$\langle x(t), y(t), z(t)\rangle = \langle 0, 1, 1\rangle + t\langle -\pi/2, 1, 1\rangle$$

The Asteroid Revisited



The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is parameterized by

$$\mathbf{r}(t) = \langle a\cos^3 t, a\sin^3 t\rangle, \qquad 0 \le t \le \pi/2$$

We found that

$$s(t) = \frac{3a}{4}(1 - \cos(2t))$$

so

$$\frac{4s}{3a} = 1 - \cos(2t)$$

Let's parametrize this curve by solving for $\cos(t)$ and $\sin(t)$.

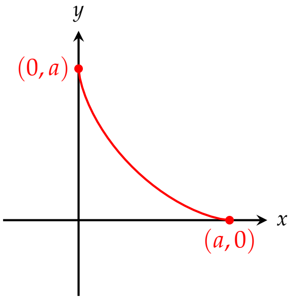
We have $\cos(2t) = 1 - 4s/3a$ so by the formulas

$$\cos(t) = \sqrt{\frac{1+\cos(2t)}{2}}, \quad \sin(t) = \sqrt{\frac{1-\cos(2t)}{2}}$$

we get $\cos(t) = \sqrt{1 - 2s/3a}, \quad \sin(t) = \sqrt{2s/3a}$ and

$$x(s) = a\left(1 - \frac{2s}{3a}\right)^{\frac{3}{2}}, \quad y(s) = a\left(\frac{2s}{3a}\right)$$

The Asteroid Revisited



The curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is parameterized by arc length:

$$x(s) = a\left(1 - \frac{2s}{3a}\right)^{\frac{3}{2}},$$

$$y(s) = a\left(\frac{2s}{3a}\right)^{\frac{3}{2}}$$

What points on the curve correspond to $s = 0$? What about $s = 3a/4$? What about $s = 3a/2$?

At $s = 0$, $(x(0), y(0)) = (a, 0)$ (the starting point)

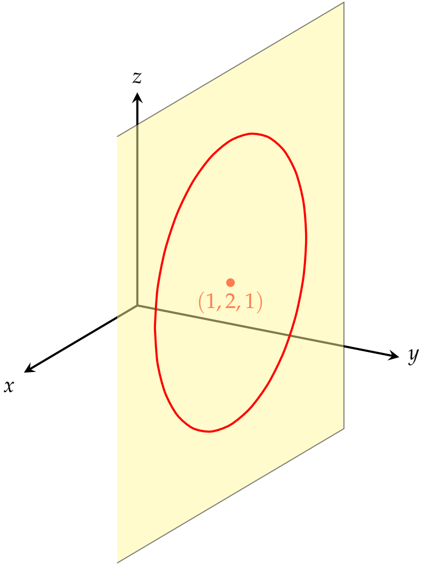
At $s = 3a/4$, $(x(3a/4), y(3a/4)) = (2^{-\frac{3}{2}}a, ^{-\frac{3}{2}}a)$ (the midpoint)

At $s = 3a/2$, $(x(3a/2), y(3a/2)) = (0, a)$ (the ending point)

What is the length of the curve from $(0, a)$ to $(a, 0)$?

The arc length is $3a/2$ since this value of s puts us at the endpoint

Going Around in Circles



A circle of radius 2 has center $(1, 2, 1)$ and lies in a plane parallel to the xz plane. Find a parameterization for the circle.

$$x(t) = 1 + 2\cos(t)$$
$$y(t) = 2$$
$$z(t) = 1 + 2\sin(t)$$

Integration Along Curves

If $f(x,y)$ is a function of two variables and C is a curve $(x(t),y(t))$ for $a \le t \le b$, then

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\,\sqrt{x'(t)^2 + y'(t)^2}\,dt$$

If $f(x,y,z)$ is a function of three variables and C is a curve $(x(t),y(t),z(t))$ with $a \le t \le b$, then

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$$

Computing the integral of a function along a curve is a *one-variable* Calculus problem!

Puzzler #1

Find $\int_C \sin x \, ds$ if C is the curve $(\operatorname{arcsec}(t), \ln(t))$ for $1 \leq t \leq \sqrt{2}$. *Hint*:

$$\frac{d}{dt} \operatorname{arcsec}(t) = \frac{1}{t\sqrt{t^2 - 1}}$$

If $\mathbf{r}(t) = \operatorname{arcsec}(t)\mathbf{i} + \ln(t)\mathbf{j}$ then $\mathbf{r}'(t) = \frac{1}{t\sqrt{t^2-1}}\mathbf{i} + \frac{1}{t}\mathbf{j}$ so

$$ds = \sqrt{\frac{1}{t^2 - 1}}$$

If $\theta = \operatorname{arcsec}(t)$ then $\cos(\theta) = \dfrac{1}{t}$ and $\sin\theta = \sqrt{1 - \dfrac{1}{t^2}} = \sqrt{\dfrac{t^2 - 1}{t^2}}$. Hence

$$\begin{aligned}\int_C \sin x \, ds &= \int_1^{\sqrt{2}} \sqrt{\frac{t^2 - 1}{t^2}} \sqrt{\frac{1}{t^2 - 1}} \, dt \\ &= \int_1^{\sqrt{2}} \frac{1}{t} \, dt \\ &= \ln\sqrt{2}.\end{aligned}$$

Puzzler #2

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$$

Suppose C is the curve from $(0,1,2)$ to $(1,2,3)$. Parametrize this curve and find $\int_C xyz\,ds$.

The parametrization is

$$(x(t),y(t),z(t)) = (t, 1+t, 2+t), \quad 0 \le t \le 1$$

so $x'(t)^2 + y'(t)^2 + z'(t)^2 = 3$. We get

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_0^1 t(1+t)(2+t)\sqrt{3}\,dt \\ &= \sqrt{3}\int_0^1 (2t+3t^2+t^3)\,dt \\ &= \frac{7\sqrt{3}}{3}\end{aligned}$$

Puzzler #3

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt$$

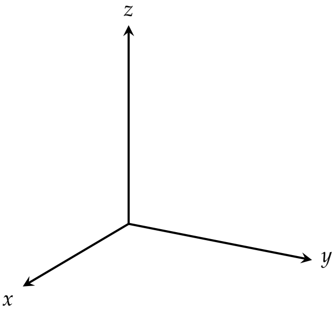
Find $\int_C f(x,y,z)\,ds$ if $f(x,y,z) = \dfrac{x+y}{y+z}$ and C is the curve

$$\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{\frac{3}{2}}\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 1.$$

Note that $x'(t) = z'(t) = 1$, $y'(t) = t^{\frac{1}{2}}$ so $ds = \sqrt{2+t}$, $\frac{x+y}{y+z} = 1$, and

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_0^1 \sqrt{2+t}\,dt \\ &= \int_2^3 \sqrt{u}\,du && \text{where } u = 2+t \\ &= \frac{2}{3}\left(3^{\frac{3}{2}} - 2^{\frac{3}{2}}\right).\end{aligned}$$

Sketching a Surface by Traces



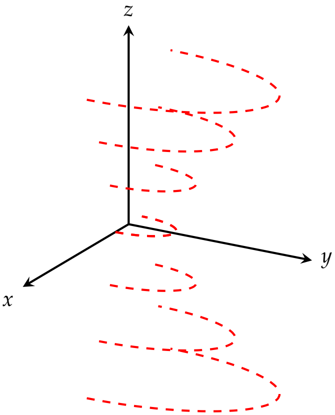
Problem: Sketch the surface

$$4x^2 + y^2 - z^2 = 1$$

- Find traces in planes $z = 0, \pm 1, \pm 2, \pm 3$
- Find traces in planes $x = 0, y = 0$

We'll cover this and other material about surfaces on Wednesda 9/13

Sketching a Surface by Traces



Problem: Sketch the surface

$$4x^2 + y^2 - z^2 = 1$$

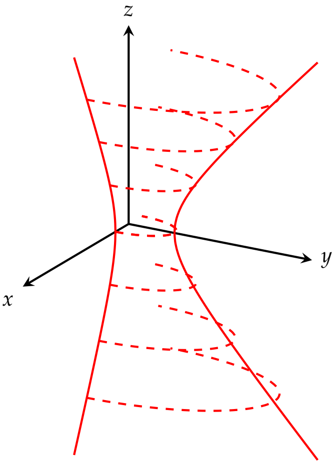
- Find traces in planes $z = 0, \pm 1, \pm 2, \pm 3$
- Find traces in planes $x = 0, y = 0$

$$4x^2 + y^2 = 1 + z^2$$

For each z, we get the equation of an ellipse

We'll cover this and other material about surfaces on Wednesda 9/13

Sketching a Surface by Traces



Problem: Sketch the surface

$$4x^2 + y^2 - z^2 = 1$$

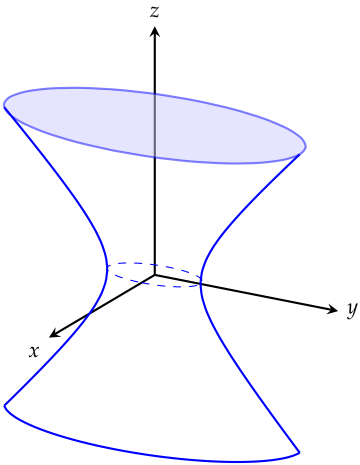
- Find traces in planes $z = 0, \pm 1, \pm 2, \pm 3$
- Find traces in planes $x = 0$, $y = 0$

In the plane $x = 0$, we get $y^2 - z^2 = 1$

In the plane $y = 0$, we get $4x^2 - z^2 = 1$

We'll cover this and other material about surfaces on Wednesda 9/13

Sketching a Surface by Traces

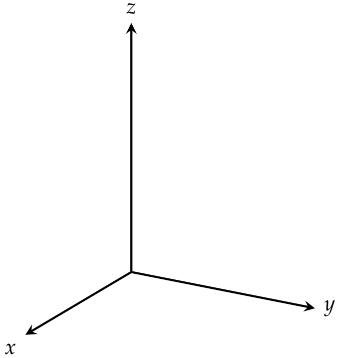


Here is the surface

$$4x^2 + y^2 - z^2 = 1$$

Note that the traces in planes parallel to the xy are ellipses and the traces in the xz and yz planes are hyperbolas

What Happens if One Sign Changes?

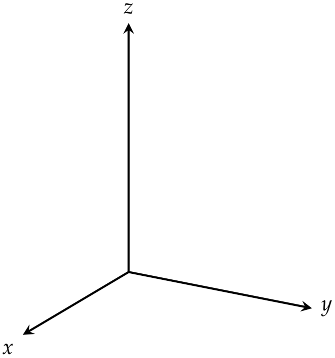


Let's try the same analysis with the surface

$$4x^2 + y^2 - z^2 = -1$$

- Find the traces in planes $z = 0, \pm 1, \pm 2, \pm 3$
- Find the traces in the xz and yz planes

What Happens if One Sign Changes?



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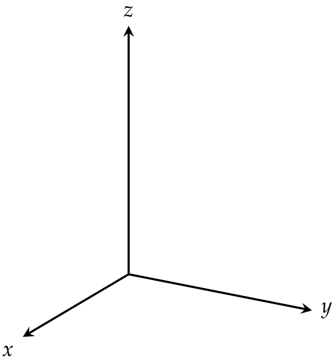
$$4x^2 + y^2 - z^2 = -1$$

- Find the traces in planes $z = 0, \pm 1, \pm 2, \pm 3$
- Find the traces in the xz and yz planes

Traces in planes $z = 0, \pm 1, \pm 2, \pm 3$:

$$4x^2 + y^2 = z^2 - 1$$

What Happens if One Sign Changes?



Let's try the same analysis with the surface

$$4x^2 + y^2 - z^2 = -1$$

- Find the traces in planes $z = 0, \pm 1, \pm 2, \pm 3$
- Find the traces in the xz and yz planes

In the xz plane, $4x^2 - z^2 = -1$

In the yz plane, $y^2 - z^2 = 1$

Reminders for the Week of September 11-15

- Homework A4 is due tonight at 11:59 PM
- Read CLP 3-1.7 for Wednesday and look at the gallery of Quadric Surfaces here
- Homework A5 is due Wednesday at 11:59 PM
- Quiz #3 on curves and reparametrizations is due on Thursday at 11:59 PM
- Read CLP 3-1.8 and CLP 3-1.9 for Friday. If you haven't already looked at the Gallery of Quadric Surfaces yet, go here now!