Math 213 - Triple Integrals - Spherical Coordinates

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March 22, 2019

Homework

- Webwork C3 on 15.7 is due tonight!
- Re-read section 15.8 and read section 15.9 for Monday
- Practice problems for 15.8 are 1-37 (odd)

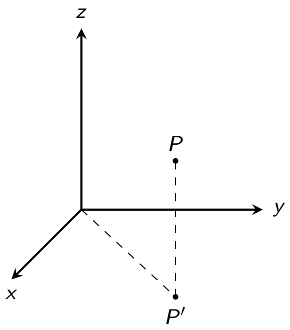
Unit III: Integral Calculus, Vector Fields

Lecture 24	Triple Integrals
Lecture 25	Triple Integrals, Continued
Lecture 26	Triple Integrals - Cylindrical Coordinates
Lecture 27	Triple Integrals - Spherical Coordinates
Lecture 28	Change of Variables for Multiple Integrals, I
Lecture 29	Change of Variable for Multiple Integrals, II
Lecture 30	Vector Fields
Lecture 31	Line Integrals (Scalar Functions)
Lecture 32	Line Integrals (Vector Functions)
Lecture 33	Fundamental Theorem for Line Integrals
Lecture 34	Green's Theorem
Lecture 35	Exam III Review

Goals of the Day

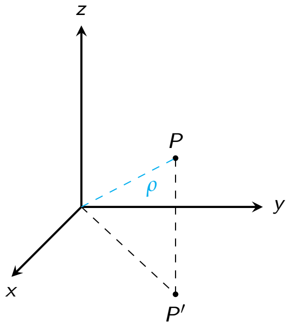
- Know how to locate points and describe regions in spherical coordinates
- Know how to evaluate triple integrals in spherical coordinates

Spherical Coordinates



The spherical coordinates (ρ, θ, ϕ) of a point P in three-dimensional space with projection P' on the xy-plane are:

Spherical Coordinates



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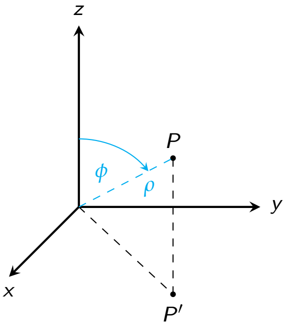
- $\rho = \sqrt{x^2 + y^2 + z^2}$, the distance $|\overrightarrow{OP}|$

Spherical Coordinates



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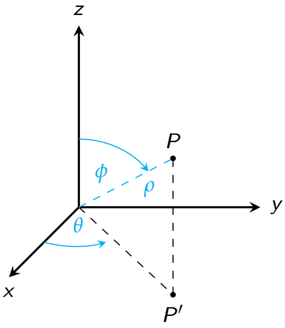
Spherical Coordinates



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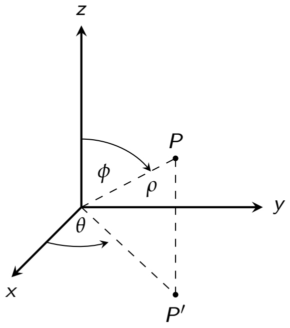
- $\rho = \sqrt{x^2 + y^2 + z^2}$, the distance $|\overrightarrow{OP}|$
- ϕ, the angle that the vector $\overrightarrow{OP}$ makes with the z-axis
- θ, the angle that the vector $\overrightarrow{OP'}$ makes with the x-axis

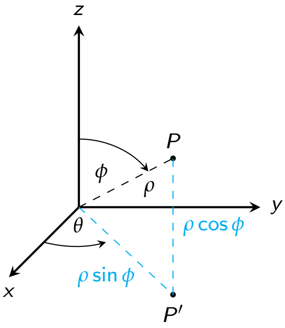
Cartesian to Spherical and Back Again

Going over:



$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\cos\phi = \frac{z}{\rho}$$

Cartesian to Spherical and Back Again



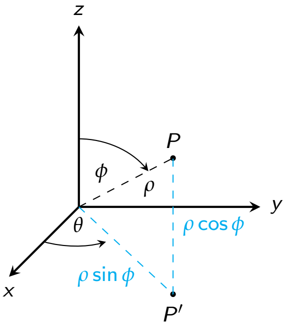
Going over:

$$\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ \tan\theta &= \frac{y}{x} \\ \cos\phi &= \frac{z}{\rho} \end{aligned}$$

Coming back:

$$\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}$$

Cartesian to Spherical and Back Again



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$$\rho = \sqrt{x^2 + y^2 + z^2}$$
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$$\cos\phi = \frac{z}{\rho}$$

Coming back:

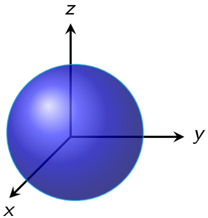
$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

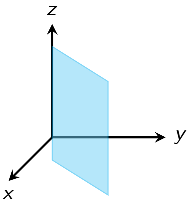
1. Find the spherical coordinates of the point $(1, \sqrt{3}, 4)$
2. Find the cartesian coordinates of the point $(4, \pi/4, \pi/2)$

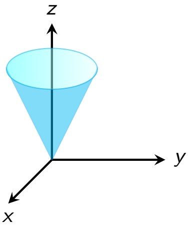
Regions in Spherical Coordinates

Match each of the following surfaces with its graph in *xyz* space

1. $\theta = c$
2. $\rho = 5$
3. $\phi = c, \quad 0 < c < \pi/2$





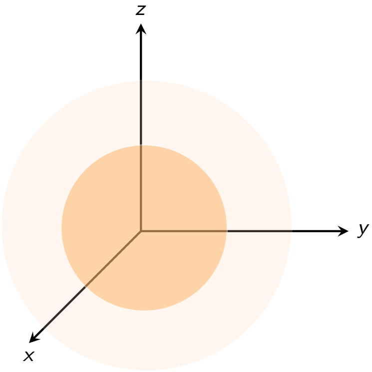


A Spherical Wedge

The region

$$E = \{(\rho, \theta, \phi) : a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

is a *spherical wedge*. What does it look like?



- $a \le \rho \le b$ means the shape lies between spheres of radius a and b

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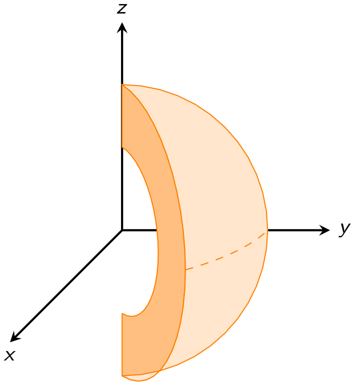

- $a \leq \rho \leq b$ means the shape lies between spheres of radius a and b
- $\alpha \leq \theta \leq \beta$ restricts the shape to a wedge-shaped region over the xy plane

A Spherical Wedge

The region

$$E = \{(\rho, \theta, \phi) : a \leq \rho \leq b,\ \alpha \leq \theta \leq \beta,\ c \leq \phi \leq d\}$$

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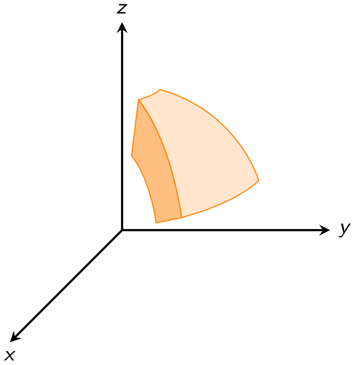


- $a \leq \rho \leq b$ means the shape lies between spheres of radius a and b
- $\alpha \leq \theta \leq \beta$ restricts the shape to a wedge-shaped region over the xy plane
- $c \leq \phi \leq d$ restricts the shape to the space between two cones about the z-axis

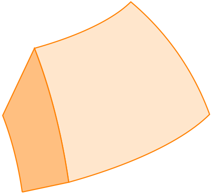
Describing Regions in Spherical Coordinates

Can you sketch each of these regions?

1. $0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi/6, \quad 0 \leq \theta \leq \pi$
2. $1 \leq \rho \leq 2, \quad \pi/2 \leq \phi \leq \pi$
3. $2 \leq \rho \leq 4, \quad 0 \leq \phi \leq \pi/3, \quad 0 \leq \theta \leq \pi$

Triple Integrals in Spherical Coordinates

We need to find the volume of a small spherical wedge



Volume comes from

$dV =$

Triple Integrals in Spherical Coordinates

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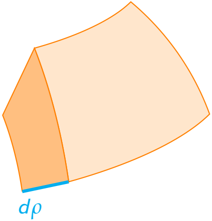


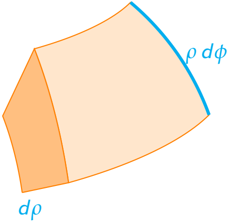
Volume comes from

- Change in ρ

$$dV = d\rho$$

Triple Integrals in Spherical Coordinates

We need to find the volume of a small spherical wedge



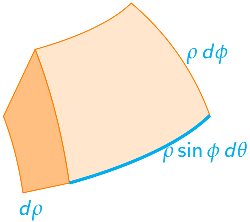
Volume comes from

- Change in ρ
- Change in ϕ

$$dV = \rho\, d\rho\, d\phi$$

Triple Integrals in Spherical Coordinates

We need to find the volume of a small spherical wedge



Volume comes from

- Change in ρ
- Change in ϕ
- Change in θ

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

Triple Integrals in Spherical Coordinates

$$\iiint_E f(x,y,z)\,dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

if E is a spherical wedge

$$E = \{(\rho,\theta,\phi) : a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

1. Find $\iiint_E y^2 z^2\,dV$ if E is the region above the cone $\phi = \pi/3$ and below the sphere $\rho = 1$
2. Find $\iiint_E y^2 dV$ if E is the solid hemisphere $x^2 + y^2 + z^2 \le 9,\ y \ge 0$
3. Find $\iiint_E \sqrt{x^2+y^2+z^2}\,dV$ if E lies above the cone $z = \sqrt{x^2+y^2}$ and between the spheres $\rho = 1$ and $\rho = 2$