Math 213 - Divergence and Curl

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Homework

- Homework D1 on 16.4 is due tonight
- Work on Stewart problems for 16.5: 1-11 (odd), 12, 13-17 (odd), 21, 23, 25
Unit IV: Vector Calculus

Lecture 36  Curl and Divergence
Lecture 37  Parametric Surfaces
Lecture 38  Surface Integrals
Lecture 39  Stokes’ Theorem
Lecture 40  The Divergence Theorem

Lecture 41  Final Review, Part I
Lecture 42  Final Review, Part II
Goals of the Day

This lecture is about two very important ‘derivatives’ of a vector field. You’ll learn:

- How to compute the \textit{curl} of a vector field and what it measures
- How to compute the \textit{divergence} of a vector field and what it measures
- (Sneak preview) The theorems that give the meaning of divergence and curl
If \( F = Pi + Qj + Rk \) is a vector field on \( \mathbb{R}^3 \), and the partial derivatives of \( P \), \( Q \), and \( R \) all exist, then the *curl* of \( F \) is a new vector field:

\[
\text{curl } F = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) i + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) j + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k
\]

This new vector field measures the “rotation” of the vector field \( F \) at a given point \((x, y, z)\):

- Its *direction* is the axis of rotation, dictated by the right-hand rule
- Its *magnitude* is the angular speed of rotation
Curl

\[
\text{curl } \mathbf{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
\]

The new vector field curl \( \mathbf{F} \) is sometimes written \( \nabla \times \mathbf{F} \) because of an easier-to-remember formula:

\[
\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \frac{\partial}{\partial x} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix}
\]
### Curl

If \( \mathbf{F} = xi + yj + zk \) then

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
  i & j & k \\
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  x & y & z
\end{vmatrix} = 0
\]

If \( \mathbf{F} = yi - xj + 0k \) then

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
  i & j & k \\
  \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
  y & -x & 0
\end{vmatrix} = -2k
\]
A gradient vector field has zero curl:

\[ \nabla \times (\nabla f) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \]

\[ = \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) i + \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) j + \left( \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) k \]

\[ = 0 \]

so the curl “detects” conservative vector fields.
The divergence of a vector field \( \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \) is a scalar function:

\[
\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

Sometimes \( \text{div } \mathbf{F} \) is written \( \nabla \cdot \mathbf{F} \):

\[
\nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\]

The divergence computes the outflow per unit volume of the vector field (thought of as a velocity field)
Divergence

If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

(same outflow at each point of space)

If $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + 0\mathbf{k}$ then

$$\nabla \cdot \mathbf{F} = \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} + 0 = 0$$

(no outflow anywhere in space)
Divergence

Remember that \( \nabla \times (\nabla f) = 0 \)? There is an analogous result for the divergence:

\[
\text{div} \; \text{curl} \; \mathbf{F} = 0
\]

You can see this using the definitions of divergence and curl:

\[
\text{div} \; \text{curl} \; \mathbf{F} = \frac{\partial}{\partial x} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)
\]

The second partial derivatives cancel in pairs by Clairaut’s theorem.

It turns out that any vector field \( \mathbf{F} \) can be written as

\[
\mathbf{F} = \nabla f + \nabla \times \mathbf{A}
\]

for a scalar potential \( f \) and a vector potential \( \mathbf{A} \).
If $f$ is a scalar function and $\mathbf{F}$ is a vector function, which of these expressions make sense? Do they define a scalar or a vector? Remember that

- $\text{curl } \mathbf{F}$ is a vector
- $\text{div } \mathbf{F}$ is a scalar

(a) $\text{curl } f$  (b) $\text{grad } f$
(c) $\text{div } \mathbf{F}$  (d) $\text{curl}(\text{grad } f)$
(e) $\text{grad } \mathbf{F}$  (f) $\text{grad}(\text{div } \mathbf{F})$
(g) $\text{div}(\text{grad } f)$  (h) $\text{grad}(\text{div } f)$
(i) $\text{curl}(\text{curl } \mathbf{F})$  (j) $\text{div}(\text{div } f)$
Conservative Vector Fields Again

Determine whether the vector field

$$\mathbf{F}(x, y, z) = y^2 z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2 z^2 \mathbf{k}$$

is conservative and, if so, find a function $f$ so that $\nabla f = \mathbf{F}$. 
Vector Identities

Show that \( \text{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \text{div} \mathbf{F} \)
Divergence Theorem, Stokes’ Theorem

**Divergence Theorem** Suppose $E$ is a simple solid region and $S$ is its boundary. Let $\mathbf{N}$ be the outward normal to $S$. Then

$$\int\int_{S} \mathbf{F} \cdot \mathbf{N} \, dS = \int\int\int_{E} \text{div} \, \mathbf{F} \, dV$$

**Stokes’ Theorem** Suppose $S$ is an oriented piecewise-smooth surface with outward normal $\mathbf{N}$, bounded by a simple closed curve $C$ with piecewise smooth boundary. Then

$$\int_{C} \mathbf{F} \cdot \, d\mathbf{r} = \int\int_{S} \text{curl} \, \mathbf{F} \cdot \mathbf{N} \, dS$$
The ‘Big Three’ Theorems of Vector Calculus

\[ F(b) - F(a) = \int_a^b F'(x) \, dx \]  
(Fundamental)

\[ \int_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA \]  
(Green)

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \, \mathbf{F} \cdot \mathbf{N} \, dS \]  
(Stokes)

\[ \iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iiint_E \text{div} \, \mathbf{F} \, dV \]  
(Divergence)