Exam 3

Name: ______________________ Section and/or TA: ______________

Last Four Digits of Student ID: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 4 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

SCORE

Multiple Choice	11	12	13	14	Total Score
50	10	15	10	15	100

Multiple Choice Questions

1. The iterated integral $\int_0^2 \int_0^{y^2} x^2 y \, dx \, dy$ has the value

   A. $16/3$

   B. $64/7$

   C. $32/7$

   D. $32/3$

   E. 10

2. Which of the following is the correct expression for $\iint_D x^2 y \, dA$ if D is the top half of the disk with center at the origin and radius 5?

   A. $\int_0^{2\pi} \int_0^5 r^3 \cos^2\theta \sin\theta \, dr \, d\theta$

   B. $\int_0^{\pi/2} \int_0^5 r^4 \cos^2\theta \sin\theta \, dr \, d\theta$

   C. $\int_0^{\pi} \int_0^5 r^3 \cos^2\theta \sin\theta \, r \, dr \, d\theta$

   D. $\int_0^{\pi} \int_{-5}^5 r^4 \cos^2\theta \sin\theta \, dr \, d\theta$

   E. $\int_0^{\pi} \int_0^5 r^3 \cos\theta \sin^2\theta \, r \, dr \, d\theta$

3. Find $\iiint_E y \, dV$ if

$$E = \{(x, y, z) : 0 \leq x \leq 3, \, 0 \leq y \leq x, \, x - y \leq z \leq x + y\}.$$

   A. 27

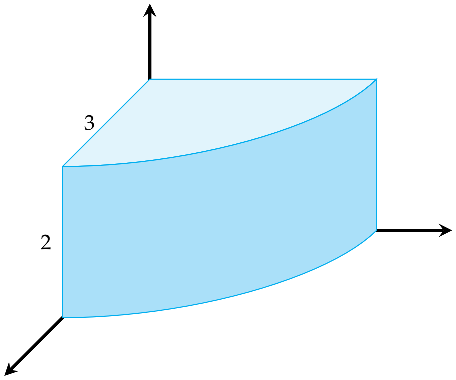
   B. $27/2$

   C. 32

   D. $32/3$

   E. 16

4. Which of the following is the correct expression for the triple integral $\iiint_E f(x, y, z)\, dV$ over the region shown?



A. $\int_0^{\pi/2} \int_0^2 \int_0^3 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$

B. $\int_0^{\pi/2} \int_0^{\pi/4} \int_0^3 f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

C. $\int_0^{\pi/2} \int_0^2 \int_0^3 f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

D. $\int_0^{\pi/2} \int_0^3 \int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$

E. $\int_0^{\pi/4} \int_0^{\pi/4} \int_0^3 f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

5. Find the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

of the transformation $x = u^2 + uv$, $y = uv^2$

None of these answers are correct! Everybody got full credit on Problem 5

A. $u^2 + uv^2$

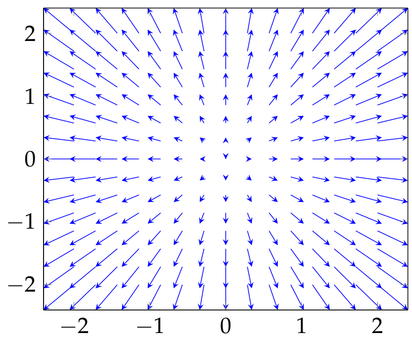
B. $4u^2 + uv^2$

C. $u^2 + u^2v$

D. $u^2 + 2uv^2$

E. $4u^2 + 2uv^2$

6. Which of the following vector fields best matches the field plot shown?



A. $y\mathbf{j}$

B. $x\mathbf{i}$

C. $x\mathbf{i} - y\mathbf{j}$

D. $x\mathbf{i} + y\mathbf{j}$

E. $y\mathbf{i} - x\mathbf{j}$

7. Find $\int_C xy^4\, ds$ if C is the right half of the circle $x^2 + y^2 = 4$

   A. $32/5$

   B. $64/5$

   C. 0

   D. $128/5$

   E. $64/3$

8. If the spherical coordinates of a point are $(\rho, \theta, \phi) = (6, \pi/3, \pi/6)$, what are its rectangular coordinates?

   A. $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 3\sqrt{3}\right)$

   B. $\left(3\sqrt{3}, \frac{3}{2}, \frac{3\sqrt{2}}{2}\right)$

   C. $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right)$

   D. $\left(\frac{3}{2}, 3\sqrt{3}, \frac{3\sqrt{3}}{2}\right)$

   E. $\left(3\sqrt{3}, \frac{3\sqrt{2}}{2}, \frac{3}{2}\right)$

9. Consider the vector field $\mathbf{F} = (2xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$. Which one of the following statements is correct?

   A. **F** is a gradient vector field $\mathbf{F} = \nabla f$ where $f(x, y) = 2x^2y^2$

   B. **F** is a gradient vector field $\mathbf{F} = \nabla f$ where $f(x, y) = x^2y + y^2x + y^3$

   C. F is a gradient vector field $\mathbf{F} = \nabla f$ where $f(x, y) = x^2y + xy^2$

   D. **F** is a gradient vector field $\mathbf{F} = \nabla f$ where $f(x, y) = x^3y + xy^2$

   E. **F** is not a conservative vector field

10. Find $\oint_C y\, dx - x\, dy$ if C is a circle with center at the origin and radius 2

   A. 8π

   B. -8π

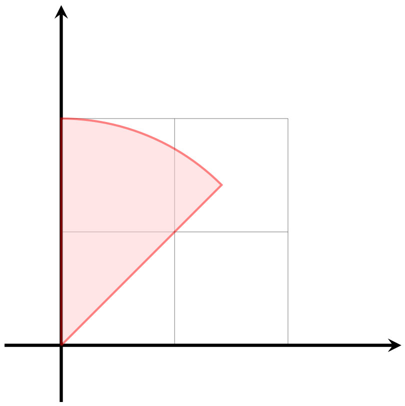
   C. 16π

   D. -16π

   E. 0

Free Response Questions

11. (10 points) This question concerns the integral $\iint_R \sin(x^2+y^2)\,dA$ where R is the region in the first quadrant enclosed by the circle $x^2+y^2=4$ and the lines $x=0$ and $y=x$.



(a) (2 points) Describe the region R in polar coordinates

Solution: $\pi/4 \leq \theta \leq \pi/2, \quad 0 \leq r \leq 2$

(b) (4 points) Set up the integral $\iint_R \sin(x^2+y^2)\,dA$ in polar coordinates

Solution: Since $r^2 = x^2+y^2$, we get

$$\iint_R \sin(x^2+y^2)\,dA = \int_{\pi/4}^{\pi/2}\int_0^2 \sin(r^2)\,r\,dr\,d\theta$$

(c) (4 points) Evaluate the integral.

Solution: Using the substitution $u = r^2$

$$\begin{aligned}\int_{\pi/4}^{\pi/2}\int_0^2 \sin(r^2)\,r\,dr\,d\theta &= \int_{\pi/4}^{\pi/2}\frac{1}{2}\int_0^4 \sin(u)\,du\,d\theta \\ &= \int_{\pi/4}^{\pi/2}\frac{1}{2}\left(1-\cos(4)\right)\,d\theta \\ &= \frac{\pi}{8}\left(1-\cos(4)\right)\end{aligned}$$

12. (15 points) This question concerns the integral $\iiint_E (x^2+y^2)\,dV$ where E is the solid in the first octant (i.e., $x \geq 0$, $y \geq 0$, $z \geq 0$) that lies under the paraboloid $z = 4 - x^2 - y^2$.

(a) (4 points) Describe the region E in cylindrical coordinates.

Solution: In the first octant, $0 \leq \theta \leq \pi/2$. The intersection of the paraboloid $z = 4 - x^2 - y^2$ with the xy plane is the cricle $x^2 + y^2 = 4$, so $0 \leq r \leq 2$. Finally, z ranges from 0 to $4 - x^2 - y^2$. In cylindrical coordinates, $4 - x^2 - y^2 = 4 - r^2$. Summing up:

$$0 \leq \theta \leq \pi/2, \quad 0 \leq r \leq 2, \quad 0 \leq z \leq 4 - r^2.$$

Scoring: 1 point each for ranges of z, r, θ, and 1 bonus point

(b) (4 points) Write $\iiint_E (x^2+y^2)\,dV$ as a triple integral in cylindrical coordinates.

Solution: In cylindrical coordinates, $x^2 + y^2 = r^2$.

$$\iiint_E (x^2+y^2)\,dV = \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} r^2\, r\,dz\,dr\,d\theta$$

Scoring: 1 point each for limits on z, r, and θ integrals, and 1 point for integrand

(c) (7 points) Evaluate the integral.

Solution:

$$\begin{aligned}
\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} \left(r^2\right) r\,dz\,dr\,d\theta &= \int_0^{\pi/2} \int_0^2 \left[r^2 z\right]\Big|_{z=0}^{z=4-r^2} r\,dr\,d\theta \\
&= \int_0^{\pi/2} \int_0^2 r^2(4-r^2)\, r\,dr\,d\theta \\
&= \int_0^{\pi/2} \frac{1}{2} \int_0^4 u(4-u)\,du\,d\theta, \qquad u = r^2 \\
&= \int_0^{\pi/2} \frac{1}{2} \left(\left[2u^2 - \frac{u^3}{3}\right]\Big|_0^4 \right) d\theta \\
&= \frac{\pi}{4}\left(2 \cdot 16 - \frac{64}{3}\right) \\
&= \frac{8\pi}{3}
\end{aligned}$$

Scoring: 2 points for each sucessful step in the threefold iterated integral, and 1 bonus point for the correct answer

13. (10 points) (a) (5 points) Find $\int_C e^y\, ds$ if C is the line segment from $(2,0)$ to $(5,4)$

Solution: Parameterize the path C as $\mathbf{r}(t) = \langle 2,0\rangle + t\langle 3,4\rangle$, $0 \le t \le 1$. Then

$$x(t) = 2+3t, \quad y(t) = 4t$$

and

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = 5dt$$

so

$$\int_C e^y\, ds = \int_0^1 e^{4t}\, 5\, dt = \frac{5}{4} e^{4t}\Big|_0^1 = \frac{5}{4}\left(e^4 - 1\right).$$

2 point for parameterization, 1 point for computation of ds, 1 point for set up of integral, 1 point for computation of integral.

(b) (5 points) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$ if $\mathbf{F} = xy^2\mathbf{i} - x^2\mathbf{j}$ and $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$ for $0 \le t \le 1$.

Solution: We are given

$$x(t) = t^3, \quad y(t) = t^2$$

so

$$x'(t) = 3t^2, \quad y'(t) = 2t.$$

Hence

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \left(t^7\mathbf{i} - t^6\mathbf{j}\right)\cdot\left(3t^2\mathbf{i} + 2t\mathbf{j}\right)\, dt\\ &= \int_0^1 \left(3t^9 - 2t^7\right)\, dt\\ &= \frac{3}{10} - \frac{1}{4}\\ &= \frac{1}{20}\end{aligned}$$

2 points for derivatives of x and y, 1 point for evaluating F along $\mathbf{r}(t)$, 2 points for computation

14. (15 points) This question concerns the integral $\iint_R x^2\,dA$ where R is the region bounded by the ellipse $9x^2 + 4y^2 = 36$.

(a) (4 points) Show that the Jacobian

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

of the transformation $x = 2u$, $y = 3v$ is 6.

Solution:

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

2 points for derivatives, 2 points for computation

(b) (4 points) Find the region S in the uv plane which is mapped to the ellipse under the transformation in part (a).

Solution: Writing $x = 2u$ and $y = 3v$, we substitute into the equation of the ellipse to find $9(4u^2) + 4(9v^2) = 36$ or $u^2 + v^2 = 1$. Hence, the region S is the region in the uv plane bounded by the circle $u^2 + v^2 = 1$.

Substitution into the equation 2 points, proper reasoning for conclusion 2 points

(c) (4 points) Using the Change of Variables formula, write $\iint_R x^2\,dA$ as an integral over S in the uv plane.

Solution: Note that $x = 2u$ and $\partial(x,y)/\partial(u,v) = 6$, so by the change of variables formula

$$\iint_R x^2\,dA = \iint_S 4u^2 \cdot 6\,du\,dv$$

Jacobian factor 2 points, rest of integrand 2 points

(d) (3 points) Evaluate the integral you found in part (c). It may be helpful to recall that

$$\cos^2\theta = \frac{1}{2}\left(1+\cos(2\theta)\right).$$

Solution: Since S is the interior of the unit circle, we can introduce polar coordinates $u = r\cos\theta$, $v = r\sin\theta$ where $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. Then

$$\begin{aligned}\iint_S 4u^2 \cdot 6\,du\,dv &= 24\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\,r\,dr\,d\theta\\ &= 24\int_0^{2\pi}\frac{1}{4}\cos^2\theta\,d\theta\\ &= 24\int_0^{2\pi}\frac{1}{4}\left(\frac{1+\cos(2\theta)}{2}\right)d\theta\\ &= 6\pi\end{aligned}$$

where we computed

$$\int_0^{2\pi}\frac{1+\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi} = \pi$$

Inner iterated integral 1 point, outer iterated integral 1 point, answer 1 point.