Math/Physics 507
Final Review

We've covered the following topics this semester:

1. Fourier Series: sine series, cosine series, orthogonal functions

2. Solution of the heat and wave equations in one-dimension by Fourier series

3. Fourier integrals: Fourier transform, inverse Fourier transform, solution of heat, wave, and Schrödinger equations

4. Special Functions and Sturm-Liouville problems associated with Partial Differential Equations:

   (a) Legendre polynomials and problems with azimuthal symmetry

   (b) Bessel functions and problems with cylindrical symmetry: wave equation in two dimensions, Fourier-Bessel series

   (c) Associated Legendre functions, spherical Bessel functions: problems with spherical symmetry

What you should know can be divided into two parts: mathematical theory and applied mathematical practice. We won't give examples here of how these functions are used to solve PDE's; we'll discuss and work problems on this in class.

1 Mathematical Theory

The underlying mathematical idea in this course is expansion in orthogonal functions. A space of functions with an inner product $f \cdot g$ is a vector space V, just like $\mathbb{R}^n$ or $\mathbb{C}^n$. The length of a vector $\mathbf{v} \in V$ is given by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

An *orthogonal basis* is a set of vectors $\{\mathbf{e}_i\}$ with the following two properties: (i) any vector in V can be expressed as a combination of the $\mathbf{e}_i$ and (ii)

$\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$. If, also, $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ for all i, the basis is *orthonormal*. Given any vector $\mathbf{v} \in V$, there are unique numbers $\{c_i\}$ so that

$$\mathbf{v} = \sum_i c_i \mathbf{e}_i$$

by property (i); to find them, we just take the dot product on both sides with $\mathbf{e}_j$ and use property (ii) to conclude that

$$c_j = \frac{\mathbf{v} \cdot \mathbf{e}_j}{\mathbf{e}_j \cdot \mathbf{e}_j}$$

Geometrically c_j is the projection of $\mathbf{v}$ onto $\mathbf{e}_j$. Note that the vectors $\mathbf{v}_N = \sum_{i=1}^N c_i \mathbf{e}_i$ solve the following approximation problem: find the linear combination of $\{\mathbf{e}_i\}_{i=1}^N$ that gives the "best fit" to $\mathbf{v}$, and that the series converges to $\mathbf{v}$ "in the mean":

$$\left\| \mathbf{v} - \sum_{i=1}^N c_i \mathbf{e}_i \right\| \to 0 \text{ as } N \to \infty.$$

For functions this is *not* the same as pointwise convergence. We also have

$$\|\mathbf{v}_N\| \leq \|\mathbf{v}\|.$$

or in other words

$$\left(\sum_{i=1}^N |c_i|^2 \right)^{1/2} \leq \|\mathbf{v}\|.$$

Where do we get a basis of orthogonal functions? In linear algebra, if A is an $N \times N$ symmetric matrix, the eigenvectors of A form an orthogonal basis. Recall that the matrix A is symmetric if for any two vectors $\mathbf{x}$ and $\mathbf{y}$,

$$(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y}).$$

Eigenvectors corresponding to distinct eigenvalues are orthogonal (you should know this argument by heart!) and one can use the Gram-Schmidt process on eigenvectors with the same eigenvalue to make these orthogonal as well.

In our case, the orthogonal functions arise as eigenvectors of a *Sturm-Liouville problem*. The simplest example of this is the problem

$$\begin{aligned} -y''(x) &= \lambda y(x) \\ y(0) &= y(1) = 0 \end{aligned}$$

for the vector space V of functions defined on $0 \le x \le 1$, equipped with the dot product

$$f \cdot g = \int_0^1 f(x)\ g(x)\ dx.$$

Here the "matrix" is replaced by the linear operator $A = -d^2/dx^2$ that maps a function y to the function $-y''$.

Exercise 1 *Show that $(Af) \cdot g = f \cdot (Ag)$ using the boundary conditions and integration by parts.*

The eigenvalue equation has solutions

$$y(x) = A \cosh \sqrt{-\lambda} x + B \sinh \sqrt{-\lambda} x$$

if $\lambda < 0$, but it is easy to see that no such function can be zero both at $x = 0$ and $x = 1$. Thus we look for solutions with $\lambda > 0$. The general solution is

$$y(x) = A \cos x\sqrt{\lambda} + B \sin x\sqrt{\lambda}$$

and the two boundary conditions $y(0) = y(1) = 0$ give $A = 0$ and $\sqrt{\lambda} = n\pi$, so the eigenvalues and eigenfunctions are

$$\lambda_n = n^2\pi^2, \quad y_n(x) = \sin(n\pi x)$$

These are orthogonal functions, and any function defined for $0 \le x \le 1$ can be expressed as a linear combination of these (sine series).

2 Applied Mathematical Practice

2.1 Fourier Series

A real-valued function periodic in $[-L, L]$ may be represented

$$f(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

The inner product is

$$f \cdot g = \int_{-L}^{L} f(x) g(x)\ dx$$

and the orthogonal functions are

$$1, \left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}, \left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty} \tag{1}$$

Fourier series converge in the mean, i.e.,

$$\left\| f - \left[\frac{A_0}{2} + \sum_{n=1}^{N}\left(A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right)\right]\right\| \to 0 \text{ as } N \to \infty$$

where

$$\|g\| = \left(\int_{-L}^{L} g(x)^2\, dx\right)^{1/2}.$$

If f has a jump discontinuity then the Fourier series for f at x converges to

$$\lim_{\varepsilon\downarrow 0}\left[\frac{1}{2}\left(f(x+\varepsilon) + f(x-\varepsilon)\right)\right]$$

Exercise 2 *Find formulas for A_n and B_n using the fact that*

$$\int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right) dx = \int_{-L}^{L}\sin^2\left(\frac{n\pi x}{L}\right) dx = L$$

Exercise 3 *If $L = \pi$ then the orthogonal functions in (1) satisfy*

$$\begin{aligned} -y''(x) &= \lambda y(x) \\ y(-\pi) &= y(\pi) \\ y'(-\pi) &= y(\pi) \end{aligned}$$

Show that if y and z solve $-y'' = \lambda y$ and $-z'' = \mu z$ and also satisfy the boundary conditions above, then

$$(\lambda - \mu)\int_{-\pi}^{\pi} y(x)z(x)\, dx = 0.$$

Use this to show that the dot product of any two distinct functions in (1) is zero.

For complex-valued periodic functions in $[-L, L]$, we may write

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{2\pi inx/L}$$

The inner product is

$$f \cdot g = \int_{-L}^{L} f(x)\, \overline{g(x)}\, dx$$

and the orthogonal functions are

$$\{\exp(2\pi inx/L)\}_{n=-\infty}^{\infty}$$

Exercise 4 *Find formulas for c_n. If f is real, express c_n in terms of the coefficients A_n and B_n in Exercise 1.*

For functions f defined on $[0, L]$ we can represent f with a Fourier sine series using the *odd extension* of f to $[-L, L]$, and we can represent f with a Fourier cosine series using the *even extension* of f. The sine functions solve the boundary value problem

$$\begin{aligned} -y'' &= \lambda y \\ y(0) &= y(L) = 0 \end{aligned}$$

while the cosine functions solve the boundary value problem

$$\begin{aligned} -y'' &= \lambda y \\ y'(0) &= y'(L) = 0 \end{aligned}$$

Fourier series are useful in solving the heat and wave equation on a bounded interval.

3 Fourier Integrals

Fourier integrals are useful in analyzing functions on the real line (such as a signal $f(t)$ which need not be periodic but dies away as $t \to -\infty$ and

$t \to +\infty$). They are a kind of "limit" of Fourier series as the period L tends to infinity. The basic formulas are

$$\begin{aligned} (\mathcal{F}f)(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \\ \left(\mathcal{F}^{-1}g\right)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx}\, dk \end{aligned}$$

We also write $\widehat{f}(k)$ for $(\mathcal{F}f)(k)$. Observe that the functions $\exp(ikx)$ solve $-y''(x) = k^2 y(x)$ and so are "eigenfunctions" but now there are a continuum of eigenfunctions rather than a discrete set. The *Fourier inversion theorem* says that

$$(\mathcal{F}^{-1}\mathcal{F}f)(x) = f(x)$$

while the *Plancherel theorem* says that

$$\|\mathcal{F}f\| = \|f\|$$

where

$$\|f\| = \left(\int_{-\infty}^{\infty} |f(x)|^2\, dx \right)$$

The *convolution theorem* says that the inverse Fourier transform of $\widehat{f}\,\widehat{g}$ is

$$\frac{1}{\sqrt{2\pi}} \int f(x-y) g(y)\, dy$$

which (up to the factor of 2π) is the *convolution of* f and g. The Fourier transform can be used to reduce the heat, wave, or Schrödinger equation for a free particle in one dimension to an algebra problem plus Fourier inversion.

Techniques for computing Fourier transforms include:

- The Calculus of residues. If f is an analytic function then Fourier integrals can be evaluated by "closing the contour" in the upper or lower half-plane to evaluate

  $$\frac{1}{2\pi i} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx.$$

  You should know how to do this and remember Jordan's Lemma on page 94 of the text.

- Shifting the contour. This is a way to evaluate the Fourier transform of a Gaussian.

4 Special Functions Associated with Boundary Value Problems

Recall the Laplacian in cylindrical and spherical coordinates:

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

(cylindrical coordinates) and

$$\begin{aligned}\nabla^2 u &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left(\nabla^2_{S^2}\right)\end{aligned}$$

where[1]

$$\nabla^2_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

(spherical coordinates). We studied the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

in two dimensions (effectively, cylindrical coordinates with the term involving z removed), the potential equation in three dimensions, and the wave equation in three dimensions. For the two wave equations, we assumed harmonic time dependence (i.e., $u(\mathbf{r}, t) = \exp(i\omega t)v(\mathbf{r})$) so we solved the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

where $k = \omega/c$. We obtained the following separation of variables solutions:

$$u(r,\theta) = J_m(kr)\left(A\cos m\theta + B\sin m\theta\right)$$

(wave equation in two dimensions)

$$u(r,\theta) = \left(Ar^\ell + Br^{-(\ell+1)}\right)P_\ell(\cos\theta)$$

[1] The notation S^2 is the mathematician's for the sphere. The "2" means that it is a two-dimensional surface.

(potential equation in three dimensions, azimuthal symmetry), and[2]

$$u(r,\theta,\phi) = [Aj_\ell(kr) + Bn_\ell(kr)]\left(C^+_{\ell m}Y^+_{\ell m}(\theta,\phi) + C^-_{\ell m}Y^-_{\ell m}(\theta,\phi)\right)$$

(wave equation, three dimensions). The special functions that arise are the following:

- Legendre polynomials $P_\ell(x)$ on $-1 \le x \le 1$, solving

  $$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dP}{dx}\right) = -\ell(\ell+1)P$$

  They are orthogonal polynomials with normalization

  $$\int_{-1}^{+1} P_\ell(x)^2 dx = \frac{2}{2\ell+1}$$

  so that any function on $-1 \le x \le 1$ can be expanded in a *Legendre series.* The Legendre Polynomials are also given by the generating function

  $$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{\ell=0}^{\infty} t^\ell P_\ell(x)$$

  and *Rodriguez' formula*

  $$P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (1-x^2)^\ell$$

- Bessel functions $J_m(x)$ of integer order, solving Bessel's equation

  $$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \left(x^2 - m^2\right)y = 0$$

  These are functions which oscillate and decay as $x \to \infty$, so their zeros α_{mn} form an infinite sequence, as do the zeros of $J'_m(x)$. They have a series representation and are also given by a generating function

  $$\exp\left((x/2)\left(t - 1/t\right)\right) = \sum_{m=-\infty}^{\infty} J_m(x)t^m$$

  We developed *Fourier-Bessel series* for the wave equation on a disc.

[2] For $m = 0$ it's understood that there is no $Y^-_{\ell 0}$. See the explicit formulas below.

- Spherical harmonics. These are built from associated Legendre functions P_ℓ^m which are given by a Rodriguez-type formula

$$P_\ell^m(x) =$$

$$\begin{aligned}
Y_{\ell 0}(\theta,\phi) &= \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta) \\
Y_{\ell m}^{+}(\theta,\phi) &= \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta)\cos m\phi \\
Y_{\ell m}(\theta,\phi) &= \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta)\sin m\phi
\end{aligned}$$

These functions are eigenfunctions for the angular part of the Laplace operator, $\nabla^2_{S^2}$. That is,

$$\nabla^2_{S^2}\left(Y_{\ell m}^{\pm}\right) = -\ell(\ell+1)Y_{\ell m}^{\pm}$$

They are also an orthonormal basis for functions on the sphere: if we denote

$$\int_{S^2} f \; d\Omega = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\sin\theta \; d\theta \; d\phi$$

and

$$f \cdot g = \int_{S^2} fg \; d\Omega$$

then

$$\begin{aligned}
Y_{\ell m}^{+} \cdot Y_{\ell' m'}^{-} &= 0 \\
Y_{\ell m}^{+} \cdot Y_{\ell' m'}^{+} &= \delta_{\ell\ell'}\delta_{mm'} \\
Y_{\ell m}^{-} \cdot Y_{\ell' m'}^{-} &= \delta_{\ell\ell'}\delta_{mm'}
\end{aligned}$$

We discussed the wave equation and Laplace's equation in spherical coordinates. If we consider functions of the form $v(r,\theta,\phi) = R(r)Y_{\ell m}^{\pm}(\theta,\phi)$ we get

$$\nabla^2 v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r} - \frac{\ell(\ell+1)}{r^2}R\right)Y_{\ell m}^{\pm}$$

so that v solves Laplace's equation if and only if

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r} - \frac{\ell(\ell+1)}{r^2}R\right) = 0$$

while v solves the Helmholtz equation if and only if

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r} - \frac{\ell(\ell+1)}{r^2}R + k^2R\right) = 0.$$

The latter equation gives rise to *spherical Bessel functions* if we set $R(r) = y(kr)$ (i.e., let $x = kr$).