Math/Physics 507 Homework 10
More Fun with Special Functions
Due Monday April 21

1. Butkov, p. 402, problem 14

2. Recall the equation for spherical Bessel functions

$$x^2 y''(x) + 2xy'(x) + \left[x^2 - l(\ell+1)\right] y = 0 \tag{1}$$

   (a) Recall that

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

   Show that $\Gamma(1/2) = \sqrt{\pi}$. Hint: let $t = u^2$ and use the result of Homework5, Problem 1

   (b) Show that, when $\ell = 0$, the change of variables $y(x) = z(x)/x$ leads to the equation

$$z'' + z = 0$$

   and conclude that

$$y_1(x) = \frac{\sin x}{x}, \quad y_2(x) = \frac{\cos(x)}{x}$$

   are two independent solutions to equation (1) when $\ell = 0$.

   (c) Show that the substitution $y(x) = u(x)/\sqrt{x}$ leads to the differential equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + \left[1 - \frac{(\ell+1/2)^2}{x^2}\right] u = 0$$

   (d) Check that the formula

$$j_0(x) = \sqrt{\frac{\pi}{2x}} J_{1/2}(x)$$

   gives $j_0(x) = \sin x/x$. Recall that

$$J_\mu(x) = \sum_{k=0}^\infty (-1)^k \frac{1}{k!\Gamma(\mu+k+1)2^{\mu+2k}} x^{\mu+2k}$$

3. Butkov, p. 403, problem 16