Math/Physics 507 Homework 10

More Fun with Special Functions

Solutions

1. The Laplacian in spherical coordinates takes the form

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

If we apply the Laplacian to a function of the form $v(r,\theta,\phi) = R(r)Y_{\ell m}(\theta,\phi)$, we get

$$\nabla^2 v = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}R\right]Y_{\ell m}(\theta,\phi),$$

so such a function will solve $\nabla^2 v = 0$ so long as $R(r)$ solves the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}R = 0.$$

This equation can be written as an Euler-type equation

$$r^2R''(r) + 2rR'(r) - \ell(\ell+1)R = 0$$

so trying $R(r) = r^\alpha$ we get the indicial equation $\alpha(\alpha-1)+2\alpha-\ell(\ell+1) = 0$, which has the roots $\alpha = \ell$ and $\alpha = -(\ell+1)$. Thus, the general solution is

$$R(r) = Ar^\ell + Br^{-(\ell+1)}. \tag{1}$$

To solve Laplace's equation $\nabla^2\varphi = 0$ in the interior of a sphere, we require that $R(r)$ be regular at the origin so $B = 0$. The general solution is a superposition

$$\varphi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\left(C_{\ell 0}Y_{\ell 0}(\theta) + \sum_{m=1}^{\infty}\left[C^+_{\ell m}Y^+_{\ell m}(\theta,\phi) + C^-_{\ell m}Y^-_{\ell m}(\theta,\phi)\right]\right)r^\ell$$

(a) The boundary condition gives

$$f(\theta,\phi) = \sum_{\ell=0}^{\infty}\left(C_{\ell 0}Y_{\ell 0}(\theta) + \sum_{m=1}^{\infty}\left[C^+_{\ell m}Y^+_{\ell m}(\theta,\phi) + C^-_{\ell m}Y^-_{\ell m}(\theta,\phi)\right]\right)a^\ell$$

so taking dot products of the equation with $Y_{\ell 0}$, $Y^{\pm}_{\ell m}$ we get the formulas (using the notation $\int_0^{2\pi}\int_0^{\pi}(\ \cdot\ )\ \sin\theta\, d\theta\, d\phi = \int\int\ (\cdot)\, d\Omega$)

$$C_{\ell 0} = a^{-\ell}\int\int f(\theta,\phi)Y_{\ell 0}(\theta,\phi)\ d\Omega$$
$$C^{+}_{\ell m} = a^{-\ell}\int\int f(\theta,\phi)Y^{+}_{\ell m}(\theta,\phi)\ d\Omega$$
$$C^{-}_{\ell m} = a^{-\ell}\int\int f(\theta,\phi)Y^{-}_{\ell m}(\theta,\phi)\ d\Omega$$

(b) Outside the sphere we require that $\varphi(r,\theta,\phi) \to 0$ as $r \to +\infty$. This means we take $A = 0$, $B = 1$ in (1) and obtain the general solution

$$\varphi(r,\theta,\phi) = \sum_{\ell=0}^{\infty}\left(C_{\ell 0}Y_{\ell 0}(\theta) + \sum_{m=1}^{\infty}\left[C^{+}_{\ell m}Y^{+}_{\ell m}(\theta,\phi) + C^{-}_{\ell m}Y^{-}_{\ell m}(\theta,\phi)\right]\right) r^{-(\ell+1)}.$$

To solve a boundary value problem

$$\nabla^2\varphi = 0$$
$$\varphi(a,\theta,\phi) = f(\theta,\phi)$$

on the exterior of the sphere, we compute the coefficients $C_{\ell 0}$ and $C^{\pm}_{\ell m}$ by the formulas

$$C_{\ell 0} = a^{\ell+1}\int\int f(\theta,\phi)Y_{\ell 0}(\theta,\phi)\ d\Omega$$
$$C^{+}_{\ell m} = a^{\ell+1}\int\int f(\theta,\phi)Y^{+}_{\ell m}(\theta,\phi)\ d\Omega$$
$$C^{-}_{\ell m} = a^{\ell+1}\int\int f(\theta,\phi)Y^{-}_{\ell m}(\theta,\phi)\ d\Omega$$

2. Recall the equation for spherical Bessel functions

$$x^2y''(x) + 2xy'(x) + \left[x^2 - \ell(\ell+1)\right]y = 0 \tag{2}$$

(a) We compute

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-1/2}e^{-t}\,dt \\ &= 2\int_0^\infty e^{-u^2}du \\ &= \int_{-\infty}^\infty e^{-u^2}du \\ &= \sqrt{\pi}\end{aligned}$$

(b) When $\ell = 0$, (2) reads

$$x^2y''(x) + 2xy'(x) + x^2y = 0. \tag{3}$$

Setting $y(x) = z(x)/x$ and using

$$\begin{aligned}y'(x) &= \frac{z'(x)}{x} - \frac{z(x)}{x^2} \\ y''(x) &= \frac{z''(x)}{x} - 2\frac{z'(x)}{x^2} + 2\frac{z(x)}{x^3}\end{aligned}$$

we get

$$xz''(x) + xz(x) = 0$$

or

$$z'' + z = 0$$

Since this equation has the independent solutions $z_1(x) = \sin x$ and $z_2(x) = \cos(x)$, we conclude that (3) has the two independent solutions

$$y_1(x) = \frac{\sin x}{x}, \quad y_2(x) = \frac{\cos(x)}{x}.$$

(c) Considering (2) in generality, we now set $y(x) = u(x)/\sqrt{x}$ and use the identities

$$\begin{aligned}y'(x) &= \frac{u'(x)}{x^{1/2}} - \frac{1}{2}\frac{u(x)}{x^{3/2}} \\ y''(x) &= \frac{u''(x)}{x^{1/2}} - \frac{u'(x)}{x^{3/2}} + \frac{3}{4}\frac{u(x)}{x^{5/2}}\end{aligned}$$

to get

$$x^{5/2}\left(\frac{u''(x)}{x^{1/2}}-\frac{u'(x)}{x^{3/2}}+\frac{3}{4}\frac{u(x)}{x^{5/2}}\right)+2x^{3/2}\left(\frac{u'(x)}{x^{1/2}}-\frac{1}{2}\frac{u(x)}{x^{3/2}}\right)+\left[x^2-\ell(\ell+1)\right]u(x)=0$$

or

$$x^2u''(x)+xu'(x)+\left[x^2-\left\{\ell(\ell+1)+\frac{1}{4}\right\}\right]u(x)=0$$

which reduces to

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left[1-\frac{(\ell+1/2)^2}{x^2}\right]u=0$$

(d) We compute

$$\begin{aligned}\sqrt{\frac{\pi}{2x}}J_{1/2}(x)&=\sqrt{\frac{\pi}{2x}}\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(1/2+k+1)2^{1/2+2k}}x^{1/2+2k}\\&=\frac{1}{x}\sqrt{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(1/2+k+1)2^{2k+1}}x^{2k+1}\end{aligned}$$

Note that

$$\begin{aligned}\Gamma(1/2+k+1)&=(1/2+k)\,\Gamma(1/2+k)\\&=(1/2+k)\,(1/2+k-1)\,\Gamma(1/2+k-1)\\&=\frac{1}{2^{k+1}}\,(2k+1)(2k-1)\cdots(1)\Gamma(1/2)\\&=\frac{\sqrt{\pi}}{2^{k+1}}(2k+1)!!\end{aligned}$$

and

$$2^kk!=(2k)(2k-2)(2k-4)\cdots(2)$$

so that

$$k!\Gamma(1/2+k+1)2^{2k+1}=\sqrt{\pi}(2k+1)!$$

It follows that

$$\begin{aligned}\sqrt{\frac{\pi}{2x}}J_{1/2}(x)&=\frac{1}{x}\sqrt{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{\sqrt{\pi}(2k+1)!}\\&=\frac{\sin x}{x}.\end{aligned}$$

3. First let's establish the "handy formula." This is really what we did in the previous problem:

$$\begin{aligned}
\Gamma\left(n+\frac{1}{2}\right) &= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) \\
&= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) \\
&= \frac{1}{2^n}(2n-1)(2n-3)\cdots(1)\sqrt{\pi} \\
&= \frac{1}{2^n}\,(2n-1)!!\sqrt{\pi}.
\end{aligned}$$

Now to work. Using the formulas

$$\begin{aligned}
J_\mu(x) &= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(\mu+k+1)2^{\mu+2k}}x^{\mu+2k} \\
j_\ell(x) &= \sqrt{\frac{\pi}{2x}}J_{\ell+1/2}(x)
\end{aligned}$$

we get

$$\begin{aligned}
j_\ell(x) &= \sqrt{\frac{\pi}{2x}}\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(\ell+k+1+1/2)2^{\ell+1/2+2k}}x^{\ell+1/2+2k} \\
&= \sqrt{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1}{2^k k!\Gamma(\ell+k+1+1/2)2^{\ell+1+k}}x^{\ell+2k} \\
&= \sqrt{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\sqrt{\pi}(2\ell+2k+1)!!}x^{\ell+2k} \\
&= \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!(2\ell+2k+1)!!}x^{\ell+2k}
\end{aligned}$$

where in the third line we used the "handy formula."