Math/Physics 507 Homework 2
Fourier Series, Part II
Solutions

1. (a) The input signal is periodic with period $2\pi/\omega$, so we take $L = \pi/\omega$. The output signal in the first case is given by

$$f_{hw}(t) = \begin{cases} 0 & -\pi/\omega \leq t \leq 0 \\ A\sin(\omega t) & 0 \leq t \leq \pi/\omega \end{cases}$$

The Fourier coefficients are then given by:

$$\begin{aligned}
a_0 &= \frac{A\omega}{\pi}\int_0^{\pi/\omega} \sin(\omega t)\ dt = \frac{A}{\pi}\left[-\cos(\omega t)\right]\Big|_0^{\pi/\omega} = \frac{2A}{\pi} \\
a_n &= \frac{A\omega}{\pi}\int_0^{\pi/\omega} \sin(\omega t)\cos(n\omega t)\ dt \\
&= \frac{A\omega}{2\pi}\int_0^{\pi/\omega} \sin\left(\left(n+1\right)\omega t\right) - \sin\left(\left(n-1\right)\omega t\right)\ dt \\
b_n &= \frac{A\omega}{\pi}\int_0^{\pi/\omega} \sin(\omega t)\sin(n\omega t)\ dt \\
&= \frac{A\omega}{2\pi}\int_0^{\pi/\omega} \cos\left((n-1)\omega t\right) - \cos\left(\left(n+1\right)\omega t\right)\ dt
\end{aligned}$$

If $n = 1$ we get

$$\begin{aligned}
a_1 &= -\frac{A\omega}{2\pi}\frac{1}{2\omega}\left[\cos(2\omega t)\right]\Big|_0^{\pi/\omega} = 0 \\
b_1 &= \frac{A\omega}{2\pi}\left[t - \frac{1}{2\omega}\sin(2\omega t)\right]\Bigg|_0^{\pi/\omega} = \frac{A}{2}
\end{aligned}$$

while for $n \neq 1$ we get

$$\begin{aligned}
a_n &= -\frac{A}{2\pi}\left[\frac{1}{n+1}\cos\left((n+1)\,\omega t\right) - \frac{1}{n-1}\cos\left((n-1)\,\omega t\right)\right]\Bigg|_0^{\pi/\omega} \\
&= \frac{A}{2\pi}\left[\frac{1}{n+1}\left(1-(-1)\right)^{(n+1)} - \frac{1}{n-1}\left[1-(-1)^{n-1}\right]\right] \\
&= -\frac{A}{2\pi}\left[\frac{2}{(n+1)\,(n-1)}\right]\left[1-(-1)^{n+1}\right] \\
&= \begin{cases} -\dfrac{2A}{\pi}\dfrac{1}{(n+1)(n-1)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{A}{2\pi}\left[\frac{1}{n-1}\sin((n-1)\omega t) - \frac{1}{n+1}\sin\left((n+1)\,\omega t\right)\right]\Bigg|_0^{\pi/\omega} \\
&= 0
\end{aligned}$$

Putting this together gives

$$f_{hw}(t) \sim \frac{A}{\pi} + \frac{A}{2}\sin(\omega t) - \frac{2A}{\pi}\sum_{n=2,4,6,\cdots}\frac{1}{(n+1)(n-1)}\cos(n\omega t)$$

which by relabelling the summation variable gives the claimed Fourier series.

(b) In this case, the output signal is

$$f_{fw}(t) = \begin{cases} -A\sin(\omega t) & -\pi/\omega \le t \le 0 \\ A\sin(\omega t) & 0 \le t \le \pi/\omega \end{cases}$$

Observe that $f_{fw}(t)$ is an even function so it must have a cosine series (all the b_n are zero). Since f_{fw} is even, we can compute the a_n (including a_0) by the formula

$$a_n = \frac{2A}{\omega}\int_0^{\pi/\omega}\sin(\omega t)\cos(n\omega t)\;dt.$$

It saves enormous time and trouble to notice that these integrals are exactly the integrals worked out for a_n in part (a) up to a factor of two. Thus, we immediately get

$$\begin{aligned} a_0 &= \frac{2A}{\pi} \\ a_n &= \begin{cases} -\frac{4A}{\pi}\frac{1}{(n+1)(n-1)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \end{aligned}$$

so that

$$f_{fw}(t) \sim \frac{2A}{\pi} - \frac{4A}{\pi} \sum_{n=2,4,6,\cdots} \frac{\cos(n\omega t)}{n^2-1}$$

as claimed.