**Math 507 Homework 3**
Fourier Series, Part III:
Solutions

1. The general solution for the wave equation describing a vibrating string with fixed ends satisfying initial conditions $u(x,0) = f(x)$ and $(\partial u/\partial t)(x,0) = g(x)$ is

$$u(x,t) = \sum_{n=1}^{\infty} \left( C_n \cos\left(\frac{n\pi ct}{L}\right) + D_n \sin\left(\frac{n\pi ct}{L}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

where

$$C_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

and

$$\frac{n\pi}{L} D_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

In this case $D_n = 0$ for all $n$ and $C_n$ is given by

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^{\xi} \frac{hx}{\xi} \sin\left(\frac{n\pi x}{L}\right) dx + \frac{2}{L}\int_{\xi}^{L} h\frac{L-x}{L-\xi} \sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= I_1 + I_2 \end{aligned}$$

Setting $u = n\pi x/L$ we get

$$\begin{aligned} I_1 &= \frac{2h}{L\xi}\left(\frac{L}{n\pi}\right)^2 \int_0^{n\pi\xi/L} u \sin u\, du \\ &= \frac{2hL}{n^2\pi^2\xi}\left[\sin\left(\frac{n\pi\xi}{L}\right) - \frac{n\pi\xi}{L}\cos\left(\frac{n\pi\xi}{L}\right)\right] \end{aligned}$$

and

$$\begin{aligned} I_2 &= \frac{2h}{L-\xi}\frac{1}{n\pi}\int_{n\pi\xi/L}^{n\pi} \left(1 - \frac{u}{n\pi}\right) \sin u\, du \\ &= \frac{2h}{L-\xi}\frac{1}{n\pi}\left(\frac{L-\xi}{L}\right)\cos\left(\frac{n\pi\xi}{L}\right) + \frac{2hL}{L-\xi}\frac{1}{n^2\pi^2}\sin\left(\frac{n\pi\xi}{L}\right) \end{aligned}$$

from which it follows that

$$\begin{aligned} C_n &= \left[\frac{2hL}{\xi} + \frac{2hL}{L-\xi}\right] \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi\xi}{L}\right) \\ &= \frac{2hL^2}{\xi(L-\xi)} \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi\xi}{L}\right) \end{aligned}$$

We can now conclude that

$$u(x,t) = \frac{2hL^2}{\xi(L-\xi)} \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \sin\left(\frac{n\pi\xi}{L}\right) \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right).$$

If we take $\xi = L/2$ we get (using that $\sin(n\pi/2)$ is zero for $n$ even, and $(-1)^{[(n-1)/2]}$ for $n$ odd)

$$u(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{8h}{n^2\pi^2} (-1)^{[(n-1)/2]} \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

which is (one hopes!) the result derived in class.

2. (Butkov, Chapter 8, bits of problems 3 and 4) A heat-conducting rod of length $L$ is thermally insulated over its surface and its ends are kept at zero temperature.

   (a) If we try a product solution of the form $p(x,t) = X(x)T(t)$ in the heat equation we get the differential equation

   $$a^{-2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

   where $\lambda$ is constant. Thus we get the ordinary differential equations

   $$\begin{aligned} T'(t) &= -a^2\lambda T(t) \\ X''(x) + \lambda X(x) &= 0 \\ X(0) &= X(L) = 0 \end{aligned}$$

   If $\lambda < 0$, say $\lambda = -\beta^2$ then $T(t) = \exp(a^2\beta^2 t)$ while $X(x) = A\cosh\beta x + B\sinh\beta x$. From the boundary conditions $X(0) = X(L) = 0$ we get that $A = 0$ and $B\sinh\beta L = 0$. Since sinh is a monotone increasing function with $\sinh(0) = 0$, the only possible solution occurs when $\beta L = 0$ so $\beta = 0$. This gives $X(x) = c$, a constant, and from the boundary conditions we infer that $X(x)$ is the zero solution. Thus, there are no nontrivial product solutions if $\lambda < 0$. If $\lambda > 0$, say $\lambda = \beta^2$, then we get

$$\begin{aligned} T(t) &= -\beta^2 a^2 T(t) \\ X''(x) + \beta^2 X(x) &= 0 \end{aligned}$$

so $T(t) = C \exp(-\beta^2 a^2 t)$ while $X(x) = A \cos \beta x + B \sin \beta x$. The boundary conditions $X(0) = X(L) = 0$ show respectively that $A = 0$ and $\sin \beta L = 0$ so that $\beta = n\pi/L$, $n = 1, 2, \cdots$ ($n = 0$ is ruled out because this gives a trivial solution). We then get a general solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} C_n \exp(-n^2\pi^2 a^2 t) \sin\left(\frac{n\pi x}{L}\right)$$

so that

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right).$$

To fit the initial condition $u(x,0) = f(x)$, we must then choose

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

(b) Taking $f(x) = u_0$ we get (setting $v = n\pi x/L$)

$$\begin{aligned} C_n &= \frac{2u_0}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right)\, dx \\ &= \frac{2u_0}{L} \frac{L}{n\pi} \int_0^{n\pi} \sin v\, dv \\ &= \frac{2u_0}{n\pi} \left(1 - (-1)^n\right) \end{aligned}$$

so that

$$C_n = \begin{cases} 0 & n \text{ even} \\ \dfrac{4u_0}{n\pi} & n \text{ odd} \end{cases}.$$

Hence

$$u(x,t) = \sum_{n=1,3,5,\cdots} \frac{4u_0}{n\pi} \exp(-n^2\pi^2 a^2 t) \sin\left(\frac{n\pi x}{L}\right).$$

This series converges beautifully when $t > 0$ but is only conditionally convergent at $t = 0$. One can compute partial sums numerically and see Gibbs' phenomenon at $x = 0$ and $x = L$ where the Fourier series is desparately trying to (a) be continuous, (b) be constant and nonzero, and (c) be zero at the endpoints. As $t \to \infty$, all the terms go to zero and the dominant term is the "heat mode"

$$\frac{4u_0}{n\pi} \exp(-\pi^2 a^2 t) \sin\left(\frac{\pi x}{L}\right).$$

3. As discussed in class Butkov 11(c) is not correct. For this reason, student solutions to 11(c) will not be graded. A correct solution will be discussed in Wednesday's (February 6) problem session, and posted on the web. Here are the solutions to parts (a) and (b) of this problem.

   (a) The heat current density is $-D\ \partial u/\partial x$ where $D$ is a diffusion coefficient. If the heat current density is held constant at $x = 0$, we get the boundary condition

   $$-D\frac{\partial u}{\partial x}(0,t) = c$$

   or

   $$\frac{\partial u}{\partial x}(0,t) = b$$

   for a constant $b$. The zero temperature condition at $x = L$ translates, as usual, to $u(L,t) = 0$, while the initial condition is clearly $u(x,0) = u_0$.

   (b) To find a steady-state solution we look for a function $v(x)$ which satisfies the heat equation (so $v''(x) = 0$), satisfies the boundary condition at $x = 0$ (so $v'(0) = b$), and satisfies the boundary condition at $x = L$ (so $v(L) = 0$). The function must be linear and therefore $v(x) = b(x - L)$.

(c) Let's outline the correct solution of the problem

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial u}{\partial x}(0,t) = b$$
$$u(L,t) = 0$$
$$u(x,0) = u_0$$

We write $u(x,t) = v(x) + w(x,t)$ where $v(x)$ is the steady-state solution found in part (b), and $w(x,t)$ is a transient solution to be computed. Substituting $u = v + w$ into the differential equation and boundary conditions above, and using the boundary conditions satisfied by $v$, we get the following homogeneous boundary value problem for the transient solution $w$:

$$\frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2}$$
$$\frac{\partial w}{\partial x}(0,t) = 0$$
$$w(L,t) = 0$$
$$w(x,0) = u_0 - v(x)$$

We look for solutions of the form $p(x,t) = X(x)T(t)$. As usual we find

$$X(x) = A\cos\beta x + B\sin\beta x$$
$$T(t) = C\exp(-a^2\beta^2 t)$$

and applying the boundary conditions we conclude that $X(L) = X'(0) = 0$. This gives the conditions

$$B = 0$$

(using the fact that $X'(x) = -\beta A\sin\beta x + \beta B\cos\beta x$ must vanish at $x = 0$) together with the eigenvalue condition

$$\cos\beta L = 0.$$

Thus

$$\beta_n = \frac{(2n+1)}{2L}\pi$$

for $n = 0, 1, 2, \cdots$. We now conclude that the general solution is given by

$$w(x,t) = \sum_{n=0}^{\infty} C_n \exp\left(-\beta_n^2 a^2 t\right) \cos\left(\beta_n x\right).$$

The functions

$$e_n(x) = \cos\left(\beta_n x\right)$$

form an orthogonal system of functions since

$$\int_0^L \cos\left(\beta_n x\right) \cos\left(\beta_m x\right)\, dx = \frac{1}{2} \int_0^L \left[\cos\left(\beta_n + \beta_m\right) x + \cos\left(\beta_n - \beta_m\right) x\right]\, dx$$

and the identities

$$\beta_n - \beta_m = (n-m)\frac{\pi}{L}$$
$$\beta_n + \beta_m = (n+m)\frac{\pi}{L}$$

hold. It easily follows that

$$\int_0^L \cos\left(\beta_n x\right) \cos\left(\beta_m x\right)\, dx = \begin{cases} L/2 & n = m \\ 0 & n \neq m \end{cases}$$

so by the hopefully-by-now-standard-expansion-in-orthogonal-functions-Yoga, we get

$$C_n = \frac{2}{L} \int_0^L f(x) \cos\left(\beta_n x\right)\, dx.$$

It remains to compute the $C_n$ when $f(x) = u_0 - v(x) = u_0 - b(x - L)$. I won't do this here, but might do it somewhere later!