

Math 507 Homework 5

Fourier Integrals, Part I

Due Monday, February 18

The Fourier transform is defined as

$$(\mathcal{F}f)(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx} dx$$

and its inverse is defined to be

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{ikx} dk$$

A reference for this material is Chapter 7 of Butkov, but note that our convention for the Fourier transform differs by a sign from his.

1. This problem concerns Gaussian integrals, which occur in quantum mechanics (for Gaussian wave packets), in probability theory (for Gaussian distributions), and in statistical mechanics.

(a) Show that

$$\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}.$$

Hint: If the integral above is I , express I^2 as an integral over x and y , and convert to polar coordinates.

(b) As shown in class, one easily obtains from this formula by scaling the formula

$$\int_{-\infty}^{\infty} \exp(-ax^2) dx = \sqrt{\frac{\pi}{a}}.$$

Show that

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2) dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}}.$$

Hint: Differentiate the identity with respect to a .

(c) Now find a formula for

$$\int_{-\infty}^{\infty} x^n \exp(-ax^2) dx$$

in terms of a and n for any positive integer n .

2. Consider the initial value problem for the heat equation for an infinite rod:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = f(x)$$

- (a) If $\widehat{u}(k, t)$ is the Fourier transform of u in the x -variable only, show that

$$\frac{\partial \widehat{u}}{\partial t} = -k^2 a^2 \widehat{u}(k, t)$$

$$\widehat{u}(k, 0) = \widehat{f}(k)$$

and solve this equation for $\widehat{u}(k, t)$.

- (b) Show that the inverse Fourier transform of $\exp(-ta^2k^2)$ is

$$\frac{1}{\sqrt{2a^2t}} \exp(-x^2/(4a^2t))$$

- (c) Apply the convolution theorem to find the general solution of the heat equation for an infinite rod.