Math/Physics 507 Homework 5

Solutions

1. (a) If $I = \int_{-\infty}^{\infty} \exp(-x^2)\, dx$ then

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-x^2)\exp(-y^2)\, dy\, dx \\ &= \int_0^{2\pi}\int_0^{\infty} \exp(-r^2) r\, dr\, d\theta \\ &= \pi \int_0^{\infty} \exp(-r^2)\, 2r\, dr \\ &= \pi \int_0^{\infty} \exp(u)\, du \\ &= \pi \end{aligned}$$

where in the last step we set $u = r^2$ and used the fact that $\int_0^{\infty} e^{-u}\, du = 1$. Thus $I^2 = \pi$ so $I = \sqrt{\pi}$.

(b) By scaling we have

$$\int_{-\infty}^{\infty} \exp(-ax^2)\, dx = \sqrt{\frac{\pi}{a}}.$$

We can apply $-d/da$ to both sides and use the fact that

$$-\frac{d}{da}\left(\exp(-ax^2)\right) = x^2 \exp(-ax^2)$$

to conclude that

$$\int_{-\infty}^{\infty} x^2 \exp(-ax^2)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

as claimed.

(c) First, note that if n is odd, the integral must be zero because the integrand is odd. Next, note that

$$\int_{-\infty}^{\infty} x^{2n} \exp(-ax^2)\, dx = \left(-\frac{d}{da}\right)^n \int_{-\infty}^{\infty} \exp(-ax^2)\, dx$$

so we need to compute

$$\left(-\frac{d}{da}\right)^n \sqrt{\frac{\pi}{a}} = \pi^{1/2}\left(-\frac{d}{da}\right)^n \left(a^{-1/2}\right).$$

This is easily (or, at least, relatively easily) seen to be

$$\pi^{1/2}\frac{1}{2}\times\frac{3}{2}\times\cdots\times\frac{2n-1}{2}a^{-(2n+1)/2}.$$

(The truly serious can try to prove this by induction!) Thus

$$\int_{-\infty}^{\infty} x^{2n}\exp(-ax^2)\,dx = \frac{(2n-1)!!}{2^n}\sqrt{\frac{\pi}{a^{2n+1}}}.$$

As a footnote, the correctly normalized Gaussian probability distribution is given by

$$dP(x) = \sqrt{\frac{a}{\pi}}\exp(-ax^2)\,dx.$$

(we divide by $\int_{-\infty}^{\infty}\exp(-ax^2)\,dx$ so that $\int_{-\infty}^{\infty} dP(x) = 1$. Also, it might be better to write $a = \sigma^{-2}$ so the distribution is

$$dP(x) = \sqrt{\frac{1}{\pi\sigma^2}}\exp(-x^2/\sigma^2)\,dx$$

In this case we get

$$\int_{-\infty}^{\infty} x^{2n}\,dP(x) = \frac{(2n-1)!!}{2^n}\sigma^{2n}$$

which is now dimensionally correct since σ has the same dimensions as x.

2. (a) The equation follows from taking Fourier transforms with respect to x on both sides. The equation

$$\frac{\partial\widehat{u}}{\partial t}(k,t) = -k^2a^2\widehat{u}(k,t)$$

has the general solution

$$\widehat{u}(k,t) = A(k)\exp(-k^2a^2t)$$

for a constant $A(k)$ depending on k. From the initial condition $\widehat{u}(k,0) = \widehat{f}(k)$ we conclude that $A(k) = \widehat{f}(k)$ and

$$\widehat{u}(k,t) = \exp(-k^2 a^2 t)\widehat{f}(k).$$

(b) The claimed formula follows from our computation of the Fourier transform of a Gaussian in class. Recall that we computed

$$\int_{-\infty}^{\infty} e^{-ikx} \exp(-\alpha x^2)\, dx = \sqrt{\frac{1}{2\alpha}} e^{-k^2/4\alpha}.$$

The integral we want is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{-a^2 k^2 t}\, dk.$$

Comparing to the in-class result, we see that the roles of x and k are reversed and $\alpha = a^2 t$ (there's also a change of sign which turns out not to matter since $\exp(-k^2 a^2 t)$ is even in k).

Either by using the in-class result or by direct computation, one gets that the inverse Fourier transform of $\exp(-a^2 k^2 t)$ is

$$\frac{1}{\sqrt{2a^2 t}} \exp(-x^2/(4a^2 t))$$

as claimed.

(c) The solution will be the convolution of f with

$$\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2a^2 t}} \exp(-x^2/4a^2 t) = \frac{1}{\sqrt{4\pi a^2 t}} \exp(-x^2/(4a^2 t)).$$

(the extra factor of $\sqrt{2\pi}$ comes from the convolution theorem). Thus

$$u(x,t) = \frac{1}{\sqrt{4\pi a^2 t}} \int_{-\infty}^{\infty} \exp(-(x-y)^2/(4a^2 t)) f(y)\, dy.$$