Math 507 Homework 6

Solutions

1. (Butkov problem 2)

   (a) Compute

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\lambda x^2}\cos(\beta x)e^{-ikx}\,dx = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} e^{-\lambda x^2}\left[e^{i(\beta-k)x}+e^{-i(\beta+k)x}\right]\,dx\right) \tag{1}$$

where we've used $\cos(\beta x) = \frac{1}{2}\left[e^{i\beta x}+e^{-i\beta x}\right]$. From the general formula

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha x^2}e^{-ikx}\,dx = \frac{1}{\sqrt{2\alpha}}e^{-k^2/4\alpha}$$

we can simplify compute the right-hand side of (1):

$$\frac{1}{2}\frac{1}{\sqrt{2\lambda}}\left(e^{-(k-\beta)^2/4\lambda}+e^{-(k+\beta)^2/4\lambda}\right).$$

   (b) First, note that

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\cos(\beta x)}{x^4+a^4}e^{-ikx}dx = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\beta x}+e^{-i\beta x}}{x^4+a^4}e^{-ikx}dx.$$

If we define

$$I(k) = \int_{-\infty}^{\infty}\frac{e^{-ikx}}{x^4+a^4}dx$$

then the Fourier transform we seek is

$$\frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[I(k-\beta)+I(k+\beta)\right] \tag{2}$$

So, it's enough to find $I(k)$. Note that the identity $I(k) = I(-k)$ holds (this is a sneaky symmetry trick: $I(-k)$ can be transformed back into $I(k)$ by making the substitution $x \mapsto -x$ in the integral). So we'll just compute $I(k)$ for $k > 0$ and then deduce the general formula from this symmetry. The integrand is an analytic function away from the zeros of $z^4 + a^4$ so we'll try to use the method of residues. Since $\exp(-ikz)$ blows up for $\mathrm{Im}(z) > 0$ if $k > 0$, we'll close the contour in the lower half-plane. The zeros of $z^4 + a^4$ are

$$z_1 = \frac{a}{\sqrt{2}}(1+i),\ z_2 = \frac{a}{\sqrt{2}}(-1+i),\ \ z_3 = -\frac{a}{\sqrt{2}}(1+i),\ z_4 = \frac{a}{\sqrt{2}}(1-i)$$

so by standard arguments we get

$$I(k) = 2\pi i \left[\mathrm{Res}_{z=z_3} F(z) + \mathrm{Res}_{z=z_4} F(z)\right]$$

where

$$F(z) = \frac{\exp(-ikz)}{z^4 + a^4}$$

We have the formula (recall from last semester!)[1]

$$\mathrm{Res}_{z=z_j} F(z) = \frac{\exp(-ikz_j)}{4z_j^3}$$

so

$$\begin{aligned}
\mathrm{Res}_{z=z_3} F(z) &= -\left(\sqrt{2}\right)^3 \frac{\exp(ika/\sqrt{2}) \exp\left(-ak/\sqrt{2}\right)}{4a^3\,(1+i)^3} \\
&= \frac{\exp(ika/\sqrt{2}) \exp\left(-ak/\sqrt{2}\right)}{4a^3}\,(1+i) \\
\mathrm{Res}_{z=z_3} F(z) &= \left(\sqrt{2}\right)^3 \frac{\exp(-ika/\sqrt{2}) \exp(-ka/\sqrt{2})}{4a^3\,(1-i)^3} \\
&= \frac{\exp(-ika/\sqrt{2}) \exp(-ka/\sqrt{2})}{4a^3}\,(-1+i)
\end{aligned}$$

Letting

$$Z = \frac{\exp(ika/\sqrt{2}) \exp\left(-ak/\sqrt{2}\right)}{4a^3}\,(1+i)$$

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[1] The formula in question states that if

$$f(z) = \frac{g(z)}{h(z)}$$

and $h(z)$ has a simple zero at $z = z_0$, then

$$\mathrm{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)}.$$

we see that

$$\begin{aligned}
I(k) &= 2\pi i\left[Z - \overline{Z}\right] \\
&= 4\pi\,\mathrm{Im}(Z) \\
&= \frac{\pi}{a^3}\exp\left(-ka/\sqrt{2}\right)\left[\sin\left(ka/\sqrt{2}\right) + \cos\left(ka/\sqrt{2}\right)\right] \\
&= \frac{\pi\sqrt{2}}{a^3}\exp\left(-ka/\sqrt{2}\right)\sin\left(\frac{\pi}{4} + ka/\sqrt{2}\right).
\end{aligned}$$

where we've used the identity

$$\sin\theta + \cos\theta = \sqrt{2}\sin\left(\pi/4 + \theta\right)$$

to combine the trigonometric functions. Since $I(k) = I(-k)$ we conclude that

$$I(k) = \frac{\pi\sqrt{2}}{a^3}\exp\left(-\left|k\right|a/\sqrt{2}\right)\sin\left(\frac{\pi}{4} + \left|k\right|a/\sqrt{2}\right).$$

Finally, recalling (2), we conclude that the desired Fourier transform is

$$\begin{aligned}
\mathcal{F}\left\{\frac{1}{x^4 + a^4}e^{-i\beta x}\right\} &= \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\pi\sqrt{2}}{a^3}\exp\left(-\left|k-\beta\right|a/\sqrt{2}\right)\sin\left(\frac{\pi}{4} + \left|k-\beta\right|a/\sqrt{2}\right) \\
&\quad + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\pi\sqrt{2}}{a^3}\exp\left(-\left|k+\beta\right|a/\sqrt{2}\right)\sin\left(\frac{\pi}{4} + \left|k+\beta\right|a/\sqrt{2}\right)
\end{aligned}$$

2. (Butkov Problem 5)

   (a) We need to compute

$$\mathcal{F}_S\left\{e^{-x}\cos(x)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos(x)\sin(kx)\;dx$$

Using the addition formula $\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$ we reexpress this as

$$\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left[\sin(1+k)x - \sin(1-k)x\right]\;dx.$$

From the formula[2]

$$\int_0^\infty e^{-x}\sin\beta x\ dx = \frac{\beta}{\beta^2+1}$$

we get

$$\frac{1}{2}\sqrt{\frac{2}{\pi}}\left[\frac{1+k}{(1+k)^2+1} - \frac{1-k}{(1-k)^2+1}\right] = \sqrt{\frac{2}{\pi}}\frac{k^3}{k^4+4}$$

(b) We compute

$$\begin{aligned}\int_0^\pi \sin(x)\sin(kx)\ dx &= \frac{1}{2}\int_0^\pi [\cos(k-1)x - \cos(k+1)x]\ dx \\ &= \frac{1}{2}\left[\frac{1}{k-1}\sin(k-1)\pi - \frac{1}{k+1}\sin(k+1)\pi\right] \\ &= \frac{1}{2}\left[\frac{1}{k-1}\{-\sin(k\pi)\} + \frac{1}{k+1}\sin(k\pi)\right] \\ &= \frac{1}{k^2-1}\sin(k\pi)\end{aligned}$$

Thus,

$$\mathcal{F}_S\left\{\begin{array}{ll}\sin(x) & 0<x<\pi \\ 0 & x>\pi\end{array}\right\} = \sqrt{\frac{2}{\pi}}\frac{1}{k^2-1}\sin(k\pi)$$

3. (Butkov Problem 15) We consider the differential equation

$$\begin{aligned}\ddot{x}(t) - \alpha^2 x(t) &= 0 \\ \dot{x}(0) &= b \\ \lim_{t\to\infty} x(t) = \lim_{t\to\infty}\dot{x}(t) &= 0\end{aligned}$$

(the last condition, on behavior at infinity, is a bit stronger than what Butkov describes, but according to my e-mail to the class you may assume this. At the end of the solution to this problem, I'll discuss why Butkov's condition and this one turn out to be equivalent, for this problem). To find the unknown function we take the Fourier cosine

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[2] You can prove this by computing the imaginary part of $\int_0^\infty e^{-x}e^{i\beta x}\ dx$. This is also a standard Laplace transform formula.

transform of both sides of the equation and use the boundary conditions to derive a new equation for

$$X(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty x(t) \cos \omega t \ dt.$$

Multiplying the equation by $\sqrt{\frac{2}{\pi}} \cos \omega t$ and integrating from 0 to ∞ we get

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \left(\ddot{x}(t) - \alpha^2 x(t) \right) \cos \omega t \ dt = 0$$

The delicate part of the computation is the integration by parts for the first term:

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \int_0^\infty \ddot{x}(t) \cos(\omega t) \ dt &= \sqrt{\frac{2}{\pi}} \ \dot{x}(t) \cos(\omega t)|_0^\infty + \sqrt{\frac{2}{\pi}} \int_0^\infty \dot{x}(t) \omega \sin \omega t \ dt \\ &= \sqrt{\frac{2}{\pi}} (-b) + \sqrt{\frac{2}{\pi}} \ x(t) \omega \sin \omega t|_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty x(t) \omega^2 \cos \omega t \ dt \\ &= \sqrt{\frac{2}{\pi}} (-b) - \omega^2 X(\omega) \end{aligned}$$

Notice that we used the two boundary conditions at infinity to get rid of boundary terms in the integration by parts. We then find that

$$\sqrt{\frac{2}{\pi}} (-b) - \omega^2 X(\omega) - \alpha^2 X(\omega) = 0$$

or

$$X(\omega) = -\sqrt{\frac{2}{\pi}} \frac{b}{\omega^2 + \alpha^2}.$$

To find $x(t)$ we now compute the inverse cosine transform:

$$x(t) = -\frac{2}{\pi} \int_0^\infty \frac{b}{\omega^2 + \alpha^2} \cos \omega t \ d\omega.$$

To compute the integral we note that the integrand is even in ω so we may write

$$\begin{aligned} x(t) &= -\frac{1}{\pi} \int_{-\infty}^\infty \frac{b}{\omega^2 + \alpha^2} \cos \omega t \ d\omega \\ &= -\operatorname{Re} \left(\frac{1}{\pi} \int_{-\infty}^\infty \frac{b}{\omega^2 + \alpha^2} e^{i\omega t} \ d\omega \right). \end{aligned}$$

The inside integral can be computed using the calculus of residues. Luckily, we can close the contour in the upper half-plane to enclose the pole at $\omega = i\alpha$. We then get

$$\begin{aligned} x(t) &= -\frac{1}{\pi}(2\pi i)\frac{b}{2i\alpha}e^{-\alpha t} \\ &= -\frac{b}{\alpha}e^{-\alpha t} \end{aligned}$$

One might have guessed this. The general solution to the differential equation is $x(t) = C_1 e^{\alpha t} + C_2 e^{-\alpha t}$. Since $x(t)$ vanishes as $t \to \infty$, this means $C_1 = 0$. We now need to adjust C_2 to fit the initial condition $\dot{x}(0) = b$. But $\dot{x}(0) = -\alpha C_2$ so we get $C_2 = -b/\alpha$, and hence $x(t) = -(b/\alpha)e^{-\alpha t}$ as we computed by the Fourier transform method.

If one really wants to use Laplace transforms, recalling that $\mathcal{L}\{x(t)\} = \int_0^\infty e^{-st}x(t)\ dt$, we can do a similar integration by parts on the $\ddot{x}(t)$ term:

$$\begin{aligned} \int_0^\infty \ddot{x}(t)e^{-st}\ dt &= \dot{x}(t)e^{-st}\Big|_0^\infty + \int_0^\infty \dot{x}(t)se^{-st}\ dt \\ &= -b + x(t)se^{-st}\Big|_0^\infty + s^2\int_0^\infty x(t)e^{-st}\ dt \\ &= -b - x(0)s + s^2X(s) \end{aligned}$$

where now $X(s)$ denotes the *Laplace* transform of $x(t)$. Notice that the term $x(0)$ shows up uninvited-we'll return to this in a moment. So, taking the Laplace transform of the equation we get

$$-b - x(0)s + s^2X(s) - \alpha^2X(s) = 0$$

or

$$X(s) = \frac{b + x(0)s}{s^2 - \alpha^2}$$

The inverse Laplace transform is then

$$x(t) = b\sinh\alpha t + x(0)\cosh\alpha t.$$

In terms of exponentials this is

$$\begin{aligned} x(t) &= \frac{b}{2}\left(e^{\alpha t} - e^{-\alpha t}\right) + \frac{x(0)}{2}\left(e^{\alpha t} + e^{-\alpha t}\right) \\ &= \frac{1}{2}\left[b + x(0)\right]e^{\alpha t} + \frac{1}{2}\left[x(0) - b\right]e^{-\alpha t}. \end{aligned}$$

To get $x(t)$ bounded as $t \to \infty$, we must choose $x(0)$ so that $b+x(0) = 0$, or $x(0) = -b$. Thus we get

$$x(t) = -be^{-\alpha t}$$

(Phew!). Notice that, once we impose the condition that $x(t)$ remains bounded as $t \to \infty$, we automatically get that $x(t)$ and its derivative both vanish as $t \to \infty$. This is why Butkov's boundary condition and the one I suggested in my e-mail are equivalent *for this problem.*