Math/Physics 507 Homework 9

Fourier-Bessel Series

Solutions

1. The differential equation has solutions $J_m(kr)$. The eigenvalue equation determines the allowed values of k.

   (a) The eigenvalue condition leads to

   $$J'_m(ka) = 0$$

   If β_{mn} denotes the nth root of $J'_m(x)$, the eigenvalue condition implies that

   $$ka = \beta_{mn}$$

   so the allowed values of k are

   $$k_{mn} = \frac{\beta_{mn}}{a}.$$

   Since the functions

   $$v_{mn}(r) = J_m\left(\frac{\beta_{mn}}{a} r\right)$$

   are eigenfunctions of a Sturm-Liouville problem, they are mutually orthogonal.

   (b) Suppose that $f(r)$ is a function on $0 \le r \le a$. In the expansion

   $$\begin{aligned} f(r) &= \sum_{n=1}^{\infty} b_n J_m\left(\frac{\beta_{mn}}{a} r\right) \\ &= \sum_{n=1}^{\infty} b_n v_{mn}(r) \end{aligned}$$

   ($m \ge 1$), we have

   $$b_n = \frac{f \cdot v_{mn}}{v_{mn} \cdot v_{mn}}$$

   where

   $$f \cdot g = \int_0^a f(r) g(r)\ r\ dr.$$

We need to work out the normalization integral

$$v_{mn} \cdot v_{mn} = \int_0^a J_m\left(\frac{\beta_{mn}}{a} r\right)^2 \; r \; dr.$$

Let's write

$$k_{mn} = \frac{\beta_{mn}}{a}.$$

Recall the formula

$$(\lambda - \mu) \int_0^a y(r) z(r) \; r \; dr = \left\{ r \left[y(r) z'(r) - y'(r) z(r) \right] \right\}\Big|_0^a \tag{1}$$

true for y and z solutions of the eigenvalue problems

$$\left[-\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right) + \frac{m^2}{r^2} \right] y = \lambda y$$
$$\left[-\frac{1}{r}\frac{d}{dr}\left(r\frac{dy}{dr}\right) + \frac{m^2}{r^2} \right] z = \mu y$$

Let's take $y(r) = J_m(kr)$ and $z(r) = J_m(k_{mn} r)$ (the same trick that was used in the notes and in the text to evaluate the normalization integral for the problem with the boundary condition $J_m(ka) = 0$). Using the facts that $y(r)$ and $z(r)$ are regular at $r = 0$ (so the term at $r = 0$ on the right-hand side of (1) is zero), that $y'(a) = kJ'_m(ka)$, and that $z'(a) = 0$ (this is exactly the eigenvalue condition), we get

$$\left(k^2 - k_{mn}^2\right) \int_0^a J_m(kr) J_m(k_{mn} r) \; r \; dr = -akJ'_m(ka) J_m(k_{mn} a).$$

Again, we'll differentiate with respect to k and set $k = k_{mn}$ to get the orthogonality relation. First, take the derivative with respect to k:

$$2k \int_0^a J_m(kr) J_m(k_{mn} r) \; r \; dr + \left(k^2 - k_{mn}^2\right) \{\cdots\} = -aJ'_m(ka) J_m(k_{mn} a) - a^2 k J''_m(ka) J_m(k_{mn} a) \tag{2}$$

where the $\{\cdots\}$ means “something I don’t have to calculate because it’s going to be zero anyway when $k = k_{mn}$.” Next, evaluate at $k = k_{mn}$ to get

$$2k_{mn}\int_0^a J_m(k_{mn}r)^2\ r\ dr = -a^2 k_{mn} J_m''(ka) J_m(k_{mn}a)$$

(note that one right-hand term in (2) disappears when $k = k_{mn}$ since $J_m'(k_{mn}a) = 0$). Thus

$$\int_0^a J_m(k_{mn}r)^2\ r\ dr = -\frac{a^2}{2} J_m''(ka) J_m(k_{mn}a). \tag{3}$$

To compute the second derivative we use the differential equation

$$x^2 J_m''(x) + x J_m'(x) + \left(x^2 - m^2\right) J_m(x) = 0$$

with $x = k_{mn}a$ to conclude that

$$k_{mn}^2 a^2 J_m''(k_{mn}a) + \left(k_{mn}^2 a^2 - m^2\right) J_m(k_{mn}a) = 0$$

or

$$J_m''(k_{mn}a) = -\left(1 - \frac{m^2}{a^2 k_{mn}^2}\right) J_m(k_{mn}a). \tag{4}$$

Combining (3) and (4) and using the fact that $\beta_{mn} = k_{mn}a$, we get

$$\int_0^a J_m(k_{mn}r)^2\ r\ dr = \frac{a^2}{2}\left(1 - \frac{m^2}{\beta_{mn}^2}\right) J_m^2(\beta_{mn}).$$

It now follows that the coefficient b_n is given by

$$\frac{f \cdot v_{mn}}{v_{mn} \cdot v_{mn}} = \frac{2}{a^2\left(1 - m^2/\beta_{mn}^2\right) J_m^2(\beta_{mn})} \int_0^r f(r) J_m(k_{mn}r)\ r\ dr$$

as advertised!

2. Now let’s study the eigenvalue problem above when $m = 0$. The Sturm-Liouville problem is

$$\begin{aligned} R''(r) + \frac{1}{r}R'(r) + \lambda r^2 R(r) &= 0 \\ R'(a) &= 0 \end{aligned}$$

(we set $m = 0$). The functions $J_0(kr)$ are solutions and the zeros of $J_0'(x)$ form a sequence $\{\beta_{0n}\}_{n=1}^{\infty}$ but with $\beta_{01} = 0$. Thus, the functions

$$v_n(r) = J_0((\beta_{0n}/a)r)$$

give nontrivial solutions of the eigenvalue problem for $n \geq 2$, but for $n = 1$ we get the zero solution. So, one of the eigenfunctions is missing.

(a) We look for a nontrivial solution when $\lambda = 0$. Thus consider

$$\begin{aligned} R''(r) + \frac{1}{r}R'(r) &= 0 \\ R'(a) &= 0 \end{aligned}$$

Setting $v(r) = R'(r)$ we get

$$v'(r) + \frac{1}{r}v(r) = 0$$

which is a first-order ODE solveable by separating variables. The solution is:

$$v(r) = C_1 r^{-1}.$$

Setting $C = 1$ and integrating we get

$$\begin{aligned} R(r) &= C_1 \log r + C_2. \\ R'(r) &= C_1 r^{-1} \end{aligned}$$

Applying the boundary condition $R'(a) = 0$ we get

$$R(r) = 1.$$

(b) We recall again the integration by parts formula for solutions of Bessel's equation (setting $m = 0$): if $y(r)$ and $z(r)$ are respective solutions of

$$\begin{aligned} y''(r) + \frac{1}{r}y'(r) + \frac{\lambda}{r^2}y(r) &= 0 \\ z''(r) + \frac{1}{r}z'(r) + \frac{\mu}{r^2}z(r) &= 0 \end{aligned}$$

then

$$(\lambda - \mu) \int_0^a y(r)z(r)\; r\; dr = \left.\{r\,[y(r)z'(r) - y'(r)z(r)]\}\right|_0^a . \qquad (5)$$

Now let $y(r) = 1$ and let $z(r) = J_0(\beta_{0n}(r/a))$ with $\beta_{0n} \neq 0$ (that is, $n \geq 2$). Thus $\lambda = 0$ and $\mu = \beta_{0n}^2$. Note that

$$\begin{aligned} y(a) &= 0,\;\; z(a) = J_0(\beta_{0n}) \\ y'(a) &= a^{-1},\;\; z'(a) = (\beta_{0n}/a)\, J_0'(\beta_{0n}). \end{aligned} \qquad (6)$$

and

$$\begin{aligned} y(0) &= 1,\;\; z(0) = 0 \\ y'(0) &= 0,\;\; z'(0) = J_0'(0) \end{aligned} \qquad (7)$$

In the right-hand side of formula (5) the boundary term $r = 0$ vanishes owing to (7), while the term at $r = a$ gives 0 owing to (6). We then get

$$-\beta_{0n}^2 \int_0^a 1 \;\cdot J_0(\beta_{0n}(r/a))\; r\; dr = 0$$

which shows that the function 1 is orthogonal to all of the functions $J_0(\beta_{0n}(r/a))$ with $\beta_{0n} \neq 0$.

(c) Now let us set

$$\begin{aligned} v_1(r) &= 1 \\ v_n(r) &= J_0(\beta_{0n}(r/a)) \end{aligned}$$

for $n \geq 2$. As always, if f is an arbitrary function on $0 \leq r \leq a$, we have the formula

$$f(r) \sim \sum_{n=1}^{\infty} c_n v_n(r)$$

where

$$c_n = \frac{f \cdot v_n}{v_n \cdot v_n}.$$

To give explicit formulas we need to work out the normalization integral

$$\begin{aligned} v_1 \cdot v_1 &= \int_0^a r\ dr \\ &= \frac{a^2}{2}. \end{aligned}$$

Hence (using the result of 1(c) for all n except $n = 1$)

$$\begin{aligned} c_1 &= \frac{2}{a^2} \int_0^a f(r)\ r\ dr \\ c_n &= \frac{2 \int_0^a f(r) J_0\left(\frac{\beta_{0n}}{a} r\right) dr}{a^2 J_0(\beta_{0n})^2} \end{aligned}$$

3. The temperature in the infinite cylinder obeys the heat equation

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u.$$

If we look for solutions of the form $u(r,t) = R(r)T(t)$ (building in the radial symmetry), it is not difficult to see that $R(r)$ solves the Bessel equation of order 0:

$$R''(r) + \frac{1}{r} R'(r) = \frac{\lambda}{r^2} R(r)$$

and correspondingly

$$T'(t) = -\lambda a^2 T(t)$$

so we get product solutions of the form

$$u(r,t) = e^{-\lambda_n a^2 t} v_n(r)$$

where

$$\begin{aligned} \lambda_0 &= 0 \\ \lambda_n &= \beta_{0n}^2 / b^2 \quad (n \geq 2) \end{aligned}$$

The general solution therefore takes the form

$$u(r,t) = c_1 v_1(r) + \sum_{n=2}^{\infty} c_n v_n(r) e^{-\lambda_n a^2 t}$$

and to fit the initial condition we must have

$$u_0(r) = \sum_{n=1}^{\infty} c_n v_n(r),$$

that is, the c_n are the coefficients of a Fourier-Bessel series for u_0. We can now get the claimed formulas by using the result of 2(c).