Math/Physics 507

Fourier Series, Continued

The Plucked String: No-Fault Quiz Answers

1. Suppose that f is a function on $[0, L]$ and define the odd extension of f to $[-L, L]$ by $f(-x) = -f(x)$. Write down the formula for the Fourier (was it sine or cosine?) series - i.e., say what the series looks like and give integral formulas for the coefficients.

   Answer: It's a sine series (the sines are, after all, odd functions). It takes the form

   $$f(x) \sim \sum_{k=1}^{\infty} a_k \sin(\pi k x/L)$$

   where

   $$a_k = \frac{2}{L} \int_0^L f(x) \sin(\pi k x/L)$$

2. Find the Fourier series for the "plucked string" function

$$f(x) = \begin{cases} \frac{2h}{L}x & 0 \le x \le L/2 \\ \\ \frac{2h}{L}(L-x) & L/2 \le x \le L \end{cases}$$

Answer: From the above we need to compute

$$\frac{2}{L}\int_0^L f(x)\sin(\pi kx/L)\,dx$$

Note that $f(x) = f(L-x)$. On the other hand, if $\phi_k(x) = \sin(k\pi x/L)$ (just to introduce a shorthand notation), the addition formula for sine tells us that

$$\phi_k(L-x) = \sin(\pi k - \pi kx/L) = (-1)^{k+1}\sin(\pi kx)$$

so $\phi_k(x)$ is *odd* under this reflection if k is even, and *even* under this reflection if k is odd. So, the integrals for a_k will be *zero* for all the even k's, and *nonzero* for all the odd k's with

$$\begin{aligned} a_k &= \frac{4}{L}\int_0^{L/2} f(x)\sin(k\pi x/L)\,dx \\ &= \frac{4}{L}\int_0^{L/2} \frac{2h}{L}x\sin(k\pi x/L)\,dx \\ &= \frac{8h}{L^2}\left(\frac{L}{k\pi}\right)^2\int_0^{k\pi/2} u\sin u\,du \end{aligned}$$

where in the last step we used the substitution $u = k\pi x/L$. This gives, for k *odd*,

$$a_k = h\frac{(-1)^{(k-1)/2}}{k^2\pi^2}$$

so that

$$f(x) \sim h\sum_{k=1,3,5,\cdots}\frac{(-1)^{(k-1)/2}}{k^2\pi^2}\sin(k\pi x/L)$$