Math/Physics 507
Practice Final Exam
Answers

1. (30 points) Short answer questions

   (a) We get
   $$\frac{1}{r}\frac{d}{dr}\left(rR'(r)\right) - \frac{m^2}{r^2}R - k^2R = 0$$

   (b) (10 points) Given that $(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ikx}e^{-x^2/2}dx = e^{-k^2/2}$, find the Fourier transform of $\exp(-ax^2)$. Setting $ax^2 = y^2/2$ (or $y = x\sqrt{2a}$)
   $$\begin{aligned}(2\pi)^{-1/2}\frac{1}{\sqrt{2a}}\int_{-\infty}^{\infty} e^{-ik(y/\sqrt{2a})}e^{-y^2/2}dy &= \frac{1}{\sqrt{2a}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(k/\sqrt{2a})y}e^{-y^2/2}dy \\ &= \frac{1}{\sqrt{2a}}\exp\left(-\left(k/\sqrt{2a}\right)^2/2\right) \\ &= \frac{1}{\sqrt{2a}}\exp\left(-k^2/4a\right)\end{aligned}$$

   (c) We have
   $$\int_{-\pi}^{\pi} x^2dx = 4\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{-\pi}^{\pi}\sin^2 nx\ dx$$
   or
   $$\frac{2\pi^3}{3} = 4\pi\sum_{n=1}^{\infty}\frac{1}{n^2}$$
   so
   $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

2. (35 points) The full initial/boundary problem is:
   $$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$
   (differential equation),
   $$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

(boundary conditions),

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = 0$$

(initial conditions).

(a) Separating variables we get

$$X''(x) + \lambda X(x) = 0$$
$$T''(t) + c^2 \lambda T(t) = 0$$

If $\lambda < 0$ then $X(x) = A \cosh x\sqrt{-\lambda} + B \sinh x\sqrt{\lambda}$ which cannot satisfy the boundary conditions. If $\lambda > 0$ then $X(x) = A \cos x\sqrt{\lambda} + B \sin x\sqrt{\lambda}$ t

(b) To study the effect of the boundary conditions we compute

$$X'(x) = -A\sqrt{\lambda} \sin\left(x\sqrt{\lambda}\right) + B\sqrt{\lambda} \cos\left(x\sqrt{\lambda}\right).$$

If $X'(0) = 0$ then $B = 0$, while if $X'(L) = 0$ we then get

$$A\sqrt{\lambda} \sin\left(L\sqrt{\lambda}\right) = 0$$

so

$$\sqrt{\lambda} = 0, n\pi/L.$$

for $n = 1, 2, \cdots$. This gives eigenvalues and eigenfunctions

$$\lambda = 0, \quad X(x) = 1$$

$$\lambda_n = n^2\pi^2/L^2,\ X(x) = \cos(n\pi x/L)$$

and correspondingly

$$T(t) = A_0$$
$$T(t) = A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)$$

(c) This gives a general solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} [A_n \cos(n\pi ct/L) + B_n \sin(n\pi ct/L)] \cos(n\pi x/L)$$

Observe that

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)$$
$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \cos(n\pi x/L)$$

so the initial conditions give $B_n = 0$ and

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L)$$

We can determine the constants by the usual method:

$$A_0 = \frac{\int_0^L f(x) \ \cdot 1 \ dx}{\int_0^L 1 \ dx} = \frac{1}{L} \int_0^L f(x) dx$$
$$A_n = \frac{\int_0^L f(x) \cos(n\pi x/L) \ dx}{\int_0^L \cos^2(n\pi x/L) \ dx} = \frac{2}{L} \int_0^L f(x) \ \cos(n\pi x/L) \ dx$$

3. (30 points)

(a) If $\psi(x,y) = X(x)Y(y)$, then

$$-X''(x)Y(y) - X(x)Y''(y) = EX(x)Y(y)$$

or

$$-\frac{X''(x)}{X(x)} - \frac{Y''(y)}{Y(y)} = E$$

so

$$-X''(x)/X(x) = \lambda_1, \quad -Y''(y)/Y(y) = \lambda_2$$

and $E = \lambda_1 + \lambda_2$.

(b) The differential equations above and the boundary conditions imply that

$$
\begin{aligned}
X''(x) + \lambda_1 X(x) &= 0 \\
X(0) &= X(L) = 0 \\
Y''(y) + \lambda_2 Y(y) &= 0 \\
Y(0) &= Y(L) = 0
\end{aligned}
$$

These boundary value problems have the same solutions:

$$
\begin{aligned}
\lambda_{1,n} &= n^2\pi^2/L^2, \quad X_n(x) = \sin(n\pi x/L) \\
\lambda_{2,m} &= m^2\pi^2/L^2, \quad Y_m(x) = \sin(m\pi y/L)
\end{aligned}
$$

(c) The allowed energy levels are

$$
E_{n,m} = \frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{L^2}
$$

for (n, m) a pair of positive integers..