Math/Physics 507

Practice Final Exam

Our Sponsor, Joseph Fourier

Your Name: ________________________________

Instructions: This is not the final exam. Were this the actual final, you would see something like the following:

> This is a closed-book, closed notes exam. There are three problems. You may refer to one sheet of notebook paper with formulas or facts related to the course. Please fully justify your solution to each problem. Unsupported answers, even if correct, will receive *no* credit. Good luck!

1. (30 points) Short answer questions

   (a) (10 points) The Laplacian in cylindrical coordinates is

   $$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

   Find the ordinary differential equation obeyed by $R(r)$ if $u(r,\theta,z) = R(r)e^{im\theta}e^{ikz}$ for an integer m and $\nabla^2 u = 0$.

   (b) (10 points) Given that $(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-ikx}e^{-x^2/2}dx = e^{-k^2/2}$, find the Fourier transform of $\exp(-ax^2)$.

(c) (10 points) Given the Fourier series $x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$ for $-\pi < x < \pi$, evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (Hint: compute the "length" of both sides). Remember that

$$\int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi.$$

2. (35 points) Using the separation of variables method, solve the wave equation for a string of length L and soundspeed c if the ends at $x = 0$ and $x = L$ are free, and at time $t = 0$

$$u(x, 0) = f(x)$$
$$\frac{\partial u}{\partial t}(x, 0) = 0$$

Be sure to: (a) state the ordinary differential equations which arise when you separate variables, (b) show how you use the boundary conditions at $x = 0$ and $x = L$, and (c) state the general solution to the partial differential equation. Your answer will be in the form of a series; give formulas to compute the coefficients of this series in terms of g.

3. (30 points) A quantum-mechanical particle in a two-dimensional box of side length L obeys Schrödinger's equation (setting $\hbar = 2m = 1$ for convenience)

$$-\nabla^2 \psi = E\psi$$

and obeys the boundary conditions

$$\psi(0, y) = \psi(L, y) = 0$$
$$\psi(x, 0) = \psi(x, L) = 0$$

(a) Using the method of separation of variables, find solutions of the form $\psi(x, y) = X(x)Y(y)$ There will be two separation constants λ_1 and λ_2.

(b) Using the boundary conditions, find the allowed values of the separation constants and the associated solutions.

(c) Find the energy eigenvalues E allowed by the boundary conditions.