Special Functions, I:
The Gamma Function

Culture Break: Adrien-Marie Legendre (1752-1833)[1] was a French mathematician who made important contributions to statistics, number theory, abstract algebra, and mathematical analysis. The notation $\Gamma(z)$ for the Gamma function is due to Legendre, and what we now call Legendre polynomials were introduced by him in his work on celestial mechanics, *Recherches sur la figure des planètes* (1785) He defended his thesis at the College Mazarin at the age of 18, and from 1775 to 1780 taught with Laplace at the École Militarie. In 1782 he wrote an essay on the motion of projectiles in resistive media that won a prize offered by the Berlin Academy of Sciences. He was appointed as an adjoint in the Académie des Sciences in 1783, filling a position which had been vacated by Laplace when the latter was promoted to from adjoint to associé. Legendre's subsequent work included works in celestial mechanics, the theory of elliptic functions, and analytic number theory. He lived through the French revolution but lost his fortune. In a letter to Jacobi discussing his personal circumstances (cited in the MacTutor article on Legendre already noted), he wrote:

> I married following a bloody revolution that had destroyed my small fortune; we had great problems and some very difficult moments, but my wife staunchly helped me to put my affairs in order little by little and gave me the tranquillity necessary for my customary work and for writing new works which have steadily increased my reputation.

Legendre identified himself entirely with his life's work. Poisson, cited in the MacTutor article, wrote:

> Our colleague has often expressed the desire that, in speaking of him, it would only be the matter of his works, which are, in fact, his entire life.

[1]Sources for this brief biographical sketch include the Wikipedia article on Legendre at `http://en.wikipedia.org/wiki/Adrien-Marie_Legendre` and the McTutor article at `http://www-groups.dcs.st-and.ac.uk/~history/Biographies/Legendre.html` by J. O'Connor and E. F. Robertson.

Among Legendre's works are his text *Éléments de Géometrie*, a text on analytic number theory, the *Théorie des Nombres*, and his work on elliptic functions, *Excercises du Calcul Intégral*. MacTutor offers the following quotation from Legendre on the subject of Fermat's contributions to number theory. The last comment is particularly interesting to read with the benefit of hindsight: Fermat's last theorem,[2] stated by Fermat in 1637, rested unproved for over 350 years, yielding finally to the efforts of Princeton mathematician Andrew Wiles in 1995.

> It is a matter for considerable regret that Fermat, who cultivated the theory of numbers with so much success, did not leave us with the proofs of the theorems he discovered. In truth, Messrs Euler and Lagrange, who have not disdained this kind of research, have proved most of these theorems, and have even substituted extensive theories for the isolated propositions of Fermat. But there are several proofs which have resisted their efforts.
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> *Recherches d'Analyse Indéterminée, Hist Acad Roy des Sciences (1785/1788) 513.*

[2] Fermat's last theorem states that the equation

$$x^n + y^n = z^n$$

has no nonzero integer solutions (x, y, z) if n is an integer larger than two.

The Gamma function is a special function of fundamental importance in mathematical physics, probablility theory, statistics, and combinatorics. It generalizes the familiar factorial function. In this worksheet we'll study the Gamma function and some its basic properties. We begin with the following definition of Γ as a function of a complex parameter ν:

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t}\, dt.$$

1. For what values of ν does this integral converge? Pay attention to the behavior of the integrand both as $t \downarrow 0$ and as $t \uparrow \infty$. It may be useful to note that if t is real then

   $$|t^\nu| = |t|^{\mathrm{Re}(\nu)}$$

   (why is this true?).

2. Using integration by parts, show that

$$\Gamma(\nu + 1) = \nu\Gamma(\nu)$$

for any ν. This is sometimes called the *functional equation* for the Gamma function. *Hint*: Note that

$$\Gamma(\nu + 1) = \int_0^\infty t^\nu e^{-t} dt$$

and use the usual integration by parts formula $\int u\ dv = uv - \int v\ du$.

3. Compute that $\Gamma(1) = \Gamma(2) = 1$ and conclude from the functional equation that $\Gamma(n + 1) = n!$ (If you know about mathematical induction, give a proof by induction!)

4. Let C be the contour in the complex plane illustrated below. Show that for ν real and z^ν defined by taking the branch of the logarithm with $0 < \arg(z) < 2\pi$, the formula

$$\int_C e^{-z} z^\nu \, dz = (e^{2\pi i\nu} - 1)\Gamma(\nu + 1)$$

holds. *Hint*: The contour can naturally be deformed to the second contour shown below.