Fourier Series

Part II: Finite Fourier Series: A Prelude to the FFT

Carl Friedrich Gauss (1777-1855)

Culture Break The following brief history of the Fast Fourier transform is taken from a short well-written review of the FFT by Daniel N. Rockmore, professor of Mathematics and Computer Science at Dartmouth College[1] Gauss's algorithm was rediscovered in a celebrated paper of Cooley and Tukey:[2]

> The first appearance of the FFT, like so much of mathematics, can be traced back to Gauss.[3] His interests were in certain astronomical calculations (a recurrent area of application of the FFT) having to do with the interpolation of asteroidal orbits from finite set of equally-spaced observations. Surely the prospect of a huge laborious hand calculation was good motivation for the development of a fast algorithm. Fewer calculations also implies less opportunity for error, and hence is also more numerically stable! Gauss observes that a Fourier series of bandwidth $N = N_1 N_2$ can be broken up into a computation of the N_2 subsampled DFTs of length N_2 which are combined as N_1 DFTs of length N_2, precisely as Cooley and Tukey explain. Gauss's algorithm was never published outside of his collected works.

[1] See `http://www.cs.dartmouth.edu/ rockmore/cse-fft.pdf`.

[2] J. W. Cooley and J. W. Tukey, An algorithm for machine calculation of complex Fourier series, Math. Comp., 19 (1965), 297-301.

[3] M. T. Heideman, D. H. Johnson and C. S. Burrus, Gauss and the history of the fast Fourier transform, Archive for History of Exact Sciences, 34 (1985), no. 3, 265-277

Consider a signal $f(t)$ sampled at N equally spaced points so the "data" is an N-component vector

$$(f(t_0), f(t_1), \cdots, f(t_{N-1}))$$

We'll write the components of this vector as

$$(x_0, x_1, \cdots, x_{N-1})$$

We can think of this vector as a function on the indices $0, 1, \cdots, N-1$ and write $x_n = x(n)$. Note that the x_n can be complex numbers.

The goal is to find a 'finite Fourier series' representation for such functions. We'll consider the following analogue of the complex exponentials $\exp(inx)$ for periodic functions: let

$$e_k(n) = \exp(2\pi ikn/N)$$

for each $k = 0, 1, \cdots, N$. That is, for each of these k, there is an N-component vector consisting of the numbers

$$(1, \exp(2\pi ik/N), \exp(4\pi ik/N), \cdots, \exp(2(N-1)\pi ik/N)$$

For example, $e_0(n)$ is easy to write out:

$$e_0 = (1, 1, \cdots, 1)$$

It's a function on the indices $0..N-1$ with constant value 1. You should check right now that the functions $e_k(n)$ are periodic in the sense that $e_k(n+N) = e_k(n)$.

It's useful to introduce the shorthand $\omega = \exp(2\pi i/N)$. Note that

$$\omega^N = 1.$$

We can then write

$$e_k(n) = \omega^{kn}$$

so that the Fourier basis consists of the vectors

$$\begin{aligned}
e_0 &= (1, 1, \cdots, 1) \\
e_1 &= (1, \omega, \omega^2, \cdots, \omega^{N-1}) \\
&\cdots \\
e_{N-1} &= (1, \omega^{N-1}, \omega^{2N-2}, \cdots, \omega^{(N-1)(N-1)})
\end{aligned}$$

The set-up is:

Vector spaces	N-tuples of vectors $\mathbf{x} = (x_0, x_1, \cdots, x_{N-1})$
Dot Product	$\mathbf{x} \cdot \mathbf{y} = \sum_{i=0}^{N-1} x_i \overline{y_i}$
Basis Vectors	$e_k(n) = \omega^{kn}$, $k = 0, 1, \cdots, N-1$

1. Show that the vectors e_k are really orthogonal: show that

$$e_j \cdot e_k = \begin{cases} 0 & j \neq k \\ N & j = k \end{cases}$$

   Hint: It may be useful to remember the formula

$$\sum_{n=0}^{M} \alpha^n = \frac{1 - \alpha^{N+1}}{1 - \alpha}$$

   and the relation $\omega^N = 1$.

2. Since the vectors $e_0, e_1, \cdots, e_{N-1}$ are orthogonal and there are N of them, they form a basis for the space, so any vector x is a linear combination of the vectors $e_0, e_1, \cdots, e_{N-1}$. Using the result of problem 1, show that

$$x = \frac{1}{N} \sum_{j=0}^{N-1} X_j e_j$$

where

$$X_j = \sum_{j=0}^{N-1} x_j \exp(-2\pi i j k / N)$$

Hint: Remember that the inner product involves a complex conjugation!

3. Show that the formula for the finite Fourier transform obtained in problem 2 can be written in matrix-vector form as

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \cdots \\ X_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-N} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2N} \\ 1 & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \cdots \\ x_{N-1} \end{bmatrix}$$