Math 575
Problem Set #12 Solutions

1. (p. 107, 1) Consider the function

$$f(x) = \begin{cases} \dfrac{x - \sin(x)}{1 - \cos(x)}, & x \neq 0 \\ \\ 0, & x = 0 \end{cases}$$

on the interval $(-\pi, \pi)$. First, we claim that f is continuous. As $1 - \cos(x) \neq 0$ and $\sin(x) \neq 0$ for $x \in (-\pi, \pi)$ and $x \neq 0$, we may apply L'Hospital's rule to compute

$$\begin{aligned} \lim_{x \to 0} \frac{x - \sin(x)}{1 - \cos(x)} &= \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x)} \\ &= \lim_{x \to 0} \frac{\sin(x)}{\cos(x)} \\ &= 0 \end{aligned}$$

which shows that f is continuous at $x = 0$. For $x \neq 0$, we have

$$f'(x) = 1 - \frac{(x - \sin(x))\sin(x)}{(1 - \cos(x))^2}.$$

We claim that $f'(x) \geq 0$ for $x \in (-2\pi, 2\pi)$. It suffices to show that $x\sin(x) - \sin^2(x) < 1 - 2\cos(x) + \cos^2(x)$ or equivalently that

$$x \sin x < 2(1 - \cos x).$$

Consider the function $g(x) = 2(1 - \cos x) - x \sin x$. We wish to show that $g(x) \geq 0$ on $(-2\pi, 2\pi)$. We have $g(-2\pi) = g(0) = g(2\pi) = 0$. Moreover $g'(x) = \sin x - x \cos x$, an odd function, has zeros when $\sin x = x \cos x$. Observe that $x = 0$ is a root, and there are unique roots in $(\pi/2, \pi)$ and $(-\pi, -\pi/2)$. Since $g''(x) = -x \sin x$ is even and nonpositive, the roots of $g'(x)$ correspond to maxima of g, and are positive since g is zero at the endpoints of the intervals. It follows that $f'(x)$ is nonnegative and f is increasing on $(-2\pi, 2\pi)$. Finally, f is an odd function and $\lim_{x \to 2\pi} f(x) = +\infty$ since the numerator tends to 2π and the denominator tends to zero through positive values. By symmetry $\lim_{x \to -2\pi} f(x) = -\infty$, and f maps $(-2\pi, 2\pi)$ onto $\mathbb{R}$.

2. (p. 112, 1(a)) Since f is continuous on $[a, b]$, we have $m \leq f(x) \leq M$ for all $x \in [a, b]$ and some numbers m and M. Hence

$$m(b - a) \leq \int_a^b f(t)\, dt \leq M(b - a)$$

or

$$m \leq \frac{\int_a^b f(t)\ dt}{b-a} \leq M.$$

By the Intermediate Value Theorem, there is a point c in (a, b) so that

$$f(c) = \frac{\int_a^b f(t)\ dt}{b-a}.$$

Inotherwords

$$\int_a^b f(t)\ dt = f(c)(b-a).$$