Math 575
Fall 2018
Solutions to Problem Set # 2

(1) (pp. 32–33, 1) (**2 points**) Suppose that $x_n \to x$ and $a \leq x_n \leq b$ for all n. By the definition of limit, for any $\varepsilon > 0$, there is a positive integer N so that $x_n > x - \varepsilon$ and $x_n < x + \varepsilon$ for all $n \geq N$. Thus $b > x - \varepsilon$ and $a < x + \varepsilon$, hence $a - \varepsilon < x < b + \varepsilon$. Since this holds for any $\varepsilon > 0$, we conclude that $a \leq x \leq b$.

(2) (pp. 32–33, 7) (**4 points**) Suppose that $\{z_n\}$ is a complex sequence with limit z_0. Taking $\varepsilon = 1$ we see that $|z_n| \leq |z_0| + 1$ for all sufficiently large n so that, in particular $\{z_n\}$ is bounded in modulus by some M. We may also assume that $|z_0| \leq M$.

(a) We estimate
$$|z_n^2 - z_0^2| \leq |z_n + z_0||z_n - z_0| \leq 2M|z_n - z_0|.$$
GIven $\varepsilon > 0$, choose N so that $|z_n - z_0| < \varepsilon/(2M)$. Then $|z_n^2 - Z_0^2| < \varepsilon$ as required.

(b) From the identity
$$z_n^k - z_0^k = (z_n - z_0)(z_n^{k-1} + z_n^{k-2}z_0 + \ldots + z_n z_0^{k-2} + z_0^{k-1})$$
we have
$$\left|z_n^k - z_0^k\right| \leq \left(kM^{k-1}\right)|z_n - z_0|.$$
Given $\varepsilon > 0$, choose N so that $|z_n - z_0| < \varepsilon/(kM^{k-1})$. Then $\left|z_n^k - z_0^k\right| < \varepsilon$ as required.

(3) (pp. 32–33, 10) (**not graded**) Let a_1 be an element of A and let b_1 be an upper bound. We will construct monotone sequences $\{a_n\}$ and $\{b_n\}$ as follows. Given b_n, an upper bound, and a_n, a number which is not an upper bound for A, let
$$c_n = \frac{1}{2}(a_n + b_n)$$
and choose
$$[a_{n+1}, b_{n+1}] = \begin{cases} [a_n, c_n] & \text{if } c_n \text{ is an upper bound of } A \\ [c_n, b_n] & \text{if } c_n \text{ is not an upper bound of } A \end{cases}$$
Let $I_n = [a_n, b_n]$. Since each interval is obtained from the last by bisection, we have $|b_n - a_n| \leq 2^{1-n}|b_1 - a_1|$. Moreover, by construction, $\{a_n\}$ is monotone nondecreasing and so has a limit, a. On the other hand, $\{b_n\}$ is monotone nonincreasing and so has a limit, b. Moreover, since
$$a_n \leq a \leq b \leq b_n,$$
we have
$$b - a \leq b_n - a_n = 2^{1-n}(b_1 - a_1),$$
from which we conclude that $a = b$.

Denote by c the common value of a and b. We claim that c is the least upper bound of A. Since $c = \lim b_n$, then for any $\varepsilon > 0$ the point $c - \varepsilon$ lies to the left of $[a_n, b_n]$ for some n. Since a_n is not an upper bound for A, then $c - \varepsilon$ is not an upper bound for A. On the other hand, we claim that

$x \le c$ for all $x \in A$. If not, there is an ε so that $c + \varepsilon \in A$. Since $b_n \to c$ there is an n so that $b_n < c + \varepsilon$. But then b_n is not an upper bound of A, contradicting the construction. Therefore c is an upper bound for A, and since $c \le c'$ for all other upper bounds of A, c is the least upper bound of A.

(4) (pp. 32–33, 16) (**4 points**) First, consider the function

$$f(x) = \frac{1}{2}\left(x + \frac{a}{x}\right).$$

Since

$$\left(x - \frac{a}{x}\right)^2 \ge 0$$

it follows that

$$\left(x + \frac{a}{x}\right)^2 \ge 4a$$

so that, on dividing by 4 and taking square roots

$$f(x) \ge \sqrt{a}.$$

If $x_1 < \sqrt{a}$, then $x_2 = f(x_1) \ge \sqrt{a}$. We may assume that $x_n \ge \sqrt{a}$ for all $n \ge 2$. We will show that $\{x_n\}_{n=2}^{\infty}$ is a monotone sequence. From that has already been proved, it is clear that $\{x_n\}$ is a bounded sequence with $x_n \ge \sqrt{a}$ for all $n \ge 2$.

We can compute

$$\begin{aligned} x_{n+1} - \sqrt{a} &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - \sqrt{a} \\ &= \frac{1}{2}\left(x_n - \sqrt{a}\right) + \frac{1}{2}\left(\frac{a}{x_n} - \sqrt{a}\right) \\ &= \frac{1}{2}\left(x_n - \sqrt{a}\right) + \frac{\sqrt{a}}{2}\left(\frac{\sqrt{a} - x_n}{x_n}\right) \\ &= \left(\frac{1}{2} - \frac{\sqrt{a}}{2x_n}\right)\left(x_n - \sqrt{a}\right). \end{aligned}$$

The inequality $x_n > \sqrt{a}$ implies that $\dfrac{\sqrt{a}}{2x_n} < 1/2$, so we conclude that

$$x_{n+1} - \sqrt{a} \le \frac{1}{2}\left(x_n - \sqrt{a}\right).$$

By induction,

$$x_{n+1} - \sqrt{a} \le \frac{1}{2^{n-1}}\left(x_n - \sqrt{a}\right), \quad n \ge 2$$

which shows that

$$\lim_{n \to \infty} (x_n - \sqrt{a}) = 0$$

or

$$\lim_{n \to \infty} x_n = \sqrt{a}.$$